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Vol. 27, No. 2, Nov.- Dec., 1953

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 yr. $3.00; 2 yrs. $5.75; 3 yrs. $8.50; 4 yrs. $11.00; 5 yrs. $13.00. Sponsoring subscriptions are $10.00; single copies 65¢. Reprints, bound, ½¢ per page plus 10¢ each, (thus 25 ten page reprints would cost $1.50 plus $2.50 or $3.25) provided your order is placed before your article goes to press.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, California.

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Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

CONTENTS

Probably the most famous unsolved problem in the world is Fermat's Last Theorem, namely that $x^n + y^n = z^n$ has no solution in integers, x, y, z, when n is greater than 2.

A great many papers on this theorem come to our desk, as an extreme, four came during the first ten days of last October. Several contain smart approaches which were not carried through, and practically all reprove certain simple, interesting, fundamental properties of this equation.

In the Jan.-Feb. issue, we will publish a sort of recapitulation of such properties.

If readers would add to this list and authors would permit us to publish their approaches even though they have not consummated them (giving the authors credit, of course) it would be possible to carry on a magazine roundtable on the problem. What do you think?

The nature of this problem is such that fruitful work may be done on it by almost anyone who has sufficient persistance.

Glenn James

OUR CONTRIBUTORS

(continued from back cover)

of two books, he is currently at work on two more, one (with W. V. Parker) on matrices and the other (with A. J. Robinson) on geometry. Dr. Eaves is actively interested in improving teaching of mathematics and is Vice-President of the Association of College Mathematics Teachers of Alabama and of the Auburn Section of A.E.A.

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THE MULTIPLICATION FORMULAS FOR THE BERNOULLI AND EULER POLYNOMIALS

L. Carlitz

1. Introduction. The Bernoulli and Euler polynomials may be defined by means of [5, Chapter 2]

$$(1.1)\qquad \frac{xe^{xt}}{e^x - 1} = \sum_{m=0}^{\infty} \frac{x^m}{m!} B_m(t), \qquad \frac{2e^{xt}}{e^x + 1} = \sum_{m=0}^{\infty} \frac{x^m}{m!} E_m(t).$$

It is well known that they satisfy

$$(1.2)\qquad \sum_{r=0}^{k-1} B_m\left(t + \frac{r}{k}\right) = k^{1-m} B_m(kt),$$

$$(1.3)\qquad \sum_{r=0}^{k-1} (-1)^r E_m\left(t + \frac{r}{k}\right) = k^{-m} E_m(kt),$$

where k is an arbitrary integer ≥ 1 in (1.2), while in (1.3) k is an arbitrary positive odd integer.

Bernoulli and Euler polynomials of order n may be defined by [5, Chapter 6]

$$(1.4)\qquad \frac{w_1 \cdots w_n x^n e^{xt}}{(e^{w_1 x} - 1)\cdots(e^{w_n x} - 1)} = \sum_{m=0}^{\infty} \frac{x^m}{m!} B_m^{(n)}(t \mid w_1, \cdots, w_n),$$

$$(1.5)\qquad \frac{2^n e^{xt}}{(e^{w_1 x} + 1)\cdots(e^{w_n x} + 1)} = \sum_{m=0}^{\infty} \frac{x^m}{m!} E_m^{(n)}(t \mid w_1, \cdots, w_n),$$

where the parameters w_i are arbitrary non-zero numbers. Generalizing (1.2) and (1.3) we now have

$$(1.6)\qquad \sum_{r_i=0}^{k-1} B_m^{(n)}\left[t + \frac{r_1 w_1 + \cdots + r_n w_n}{k}\right] = k^{n-m} B_m^{(n)}(kt),$$

$$(1.7)\qquad \sum_{r_i=0}^{k-1} (-1)^{r_1 + \cdots + r_n} E_m^{(n)}\left[t + \frac{r_1 w_1 + \cdots + r_n w_n}{k}\right] = k^{-m} E_m^{(n)}(kt),$$

with k odd in (1.7). Of particular interest is the case $w_1 = \cdots = w_n = 1$; (1.6) and (1.7) become

$$(1.6)' \qquad \sum_{r=o}^{n(k-1)} C_r B_m^{(n)}\left(t + \frac{r}{k}\right) = k^{n-m} B_m^{(n)}(kt),$$

$$(1.7)' \qquad \sum_{r=o}^{n(k-1)} (-1)^r C_r E_m^{(n)}\left(t + \frac{r}{k}\right) = k^{-m} E_m^{(n)}(kt),$$

where C_r is the coefficient of x^r in the expansion of $(1 + x + \cdots + x^{k-1})^n$.

It was pointed out by Nielsen [4, page 54] that $B_m(t)$ is uniquely determined by (1.2) and that $E_m(t)$ is uniquely determined by (1.3). More precisely if (1.2) holds for a single value of $k > 1$, then the set of polynomials $B_m(t)$ is completely determined, similarly if (1.3) holds for a single odd value of $\kappa > 1$.

In the present note we consider the following more general situation. Let k be a fixed integer > 1 and let $\alpha_{k1}, \cdots, \alpha_{kk}$ denote (complex) numbers such that $\alpha_{k1} + \cdots + \alpha_{kk} = 1$; let $|\lambda_k| \neq 1$ or 0 and let $\beta_{k1}, \cdots, \beta_{kk}$ be distinct numbers. Then consider the functional equation

$$(1.8) \qquad \sum_{r=1}^{k} \alpha_{kr} f_m(t + \beta_{kr}) = \lambda_k^{-m} f_m(\lambda_k t),$$

where $f_m(t)$ denotes a normalized polynomial of degree m (that is a polynomial with highest coefficient 1). We shall show that $f_m(t)$ is completely determined by (1.8); moreover, the $f_m(t)$ form an Appell set of polynomials.

2. If we put

$$f_m(t) = \sum_{j=o}^{m} C_{mj} t^{m-j} \qquad (C_{mo} = 1),$$

and compare coefficients on both sides of (1.8), we readily see that the C_{mj} are uniquely determined. Thus for $j = 1$, the coefficient of t^{m-1} in the left member is C_{m1} + terms independent of the C_{mj}, while in the right member the coefficient is precisely $\lambda_k^{-1} \neq 1$. Proceeding in this manner we determine successively $C_{m1}, C_{m2}, \cdots, C_{mm}$. Hence (1.8) is satisfied by a unique normalized polynomial $f_m(t)$ of degree m.

In the next place, differentiating both members of (1.8) with respect to t, we get

$$(2.1) \qquad \sum_{r=1}^{k} \alpha_{kr} m^{-1} f_m'(t + \beta_{kr}) = m^{-1} \lambda_k^{1-m} f_m'(\lambda_k t).$$

Since (2.1) is of the same form as (1.8) it follows from the uniqueness property that

$$f_m'(t) = mf_{m-1}(t). \tag{2.2}$$

We now recall that by an Appell set [1] of polynomials $\{f_m(t)\}$ is meant a set of polynomials satisfying (1.6). As is well known such a set is completely determined by an infinite sequence of numbers $A_0 = 1,\ A_1,\ A_2,\ \cdots$; indeed,

$$f_m(t) = \sum_{r=0}^{m} \binom{m}{r} A_{m-r} t^r, \tag{2.3}$$

and conversely; (2.3) is sometimes written in the form

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} f_m(t) = e^{xt}\Phi(x), \qquad \Phi(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} A_m. \tag{2.4}$$

We remark that the question of convergence in (2.4) is irrelevant provided we interpret the series as "formal" power series. It is an immediate consequence of (2.3) that

$$f_m(t + u) = \sum_{r=0}^{m} \binom{m}{r} t^r f_{m-r}(u);$$

we also note that $f_m(0) = A_m$.

Returning to (1.8), it is easy to write down a recursion formula for the A_m. Indeed if we take $t = 0$ in (1.8), we get

$$\sum_{r=1}^{k} \alpha_{kr} f_m(\beta_{kr}) = \lambda_k^{-m} A_m. \tag{2.5}$$

Since the coefficient of A_m on the left is 1 and on the right λ_k^{-m}, it is clear that the A_m are uniquely determined. We have therefore proved

Theorem 1. *Let k be a fixed integer > 1; $\alpha_{k1}, \cdots, \alpha_{kk}$ complex numbers such that $\alpha_{k1} + \cdots + \alpha_{kk} = 1$; $|\lambda_k| \neq 1$ or 0; $\beta_{k1}, \cdots, \beta_{kk}$ distinct. Then the equation (1.8) is satisfied by a unique set of normalized polynomials $\{f_m(t)\}$ which form an Appell set.*

It would evidently be enough to assume that $\lambda_k \neq 0$ or a root of unity.

3. It should be pointed out that not every Appell set satisfies an equation of the form (1.8). To see this we apply (1.8) to (2.4), which yields

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} \lambda_k^{-m} f_m(\lambda_k t) = e^{xt}\Phi(x) \sum_{r=1}^{k} \alpha_{kr} e^{\beta_{kr} x}. \tag{3.1}$$

Since by (2.4) the left member of (3.1)

$$= e^{xt}\Phi(\lambda_k^{-1}x),$$

we see that (3.1) implies

$$(3.2)\qquad \frac{\Phi(\lambda_k^{-1}x)}{\Phi(x)} = \sum_{r=1}^{k} \alpha_{kr} e^{\beta_{kr}x}.$$

The function $\Phi(x) = e^{x^2}$, for example, does not satisfy (3.2) for any admissible value of the parameters. It follows that the Hermite polynomials do not satisfy any relation of the form (1.8)

Incidentally (3.2) is equivalent to (2.5). More precisely, given the right member of (3.2) and $|\lambda_k| \neq 1$, then $\Phi(x)$ is uniquely determined and therefore also the set $\{f_m(x)\}$. Indeed if we put the right member of (3.2) $= \sum_0^\infty D_m x^m/m!$, $D_0 = 1$, then (3.2) is equivalent to

$$\lambda_k^{-m} A_m = \sum_{r=0}^{m} \binom{m}{r} A_r D_{m-r} \qquad (m = 1, 2, \cdots),$$

from which the A_m are uniquely determined. We may state the following theorem, which is in a sense a restatement of Theorem 1:

Theorem 2. *Given α_{kr}, β_{kr}, λ_k satisfying the hypothesis of Theorem 1. Then $\Phi(x)$ is uniquely determined by (3.2).*

If we replace $\Phi(x) = e^{\delta x}\Phi(x)$ and x by γx then it is clear from (3.2) that δ and γ can be so chosen that $B_{k1} = 0$ while β_{k2} takes on any assigned non-zero value. Hence it can be assumed that (1.8) has such a normalized form. In particular for $k = 2$ we may take

$$\alpha f_m(t) + (1 - \alpha) f_m(t + \frac{1}{2}) = \lambda^{-m} f_m(\lambda t),$$

which reduces to (1.2) for $\alpha = \frac{1}{2}$, $\lambda = 2$.

4. The multiplication formulas (1.6) and (1.7) for the Bernoulli and Euler polynomials of higher order together with the generating function in (1.4) and (1.5) suggest the following. Let $\{f_m(t)\}$ denote the set of polynomials satisfying (1.8) and let $\{g_m(t)\}$ denote the set of polynomials satisfying another equation of the form (1.8); in particular by (2.4)

$$(4.1)\qquad \sum_{m=0}^{\infty} \frac{x^m}{m!} g_m(t) = e^{xt}\Psi(x), \qquad \Psi(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} A'_m.$$

Next define a set $\{h_m(t)\}$ by means of

$$(4.2)\qquad \sum_{m=k}^{\infty} \frac{x^m}{m!} h_m(t) = e^{xt}\Phi(x)\Psi(x).$$

We wish to examine the polynomial of degree m:

$$H_m(t) = \sum_{r=1}^{k} \sum_{s=1}^{h} \alpha_{kr}\alpha'_{hs}h_m(t + \beta_{kr} + \beta'_{hs}),$$

where

$$\sum_{s=1}^{h} \alpha'_{hs}g_m(t + \beta'_{hs}) = \lambda_h'^{-m}g_m(\lambda'_h t), \tag{4.3}$$

say, is a multiplication formula satisfied by $g_m(t)$. Clearly (4.2) yields

$$\sum_{m=o}^{\infty} \frac{x^m}{m!} H_m(t) = e^{xt}\Phi(x)\Psi(x) \sum_{r,s} \alpha_{kr}\alpha'_{hs}e^{(\beta_{kr} + \beta'_{hs})x}$$

$$= e^{xt}\Phi(\lambda_k^{-1}x)\Psi(\lambda_h^{-1}x),$$

by (3.2). If we now assume $\lambda = \lambda_k = \lambda'_h$ it follows using (4.2) that

$$\sum_{m=o}^{\infty} \frac{x^m}{m!} H_m(t) = \sum_{m=o}^{\infty} \frac{x^m}{m!} \lambda^{-m}h_m(\lambda t),$$

which **implies** $H_m(t) = \lambda^{-m}h_m(\lambda t)$. This proves

Theorem 3. *Let* $\{f_m(t)\}$ *denote the solution of* (1.8) *and* $\{g_m(t)\}$ *the solution of* (4.3) *and define* $h_m(t)$ *by* (4.2). *Then if* $\lambda_k = \lambda'_h = \lambda$ *we have*

$$\sum_{r=1}^{k} \sum_{s=1}^{h} \alpha_{kr}\alpha'_{hs}h_m(t + \beta_{kr} + \beta'_{hs}) = \lambda^{-1}h_m(\lambda t). \tag{4.4}$$

Equation (4.4) is again of the form (1.8) and may be described as formed by *composition* of (1.8) and (4.3). Thus (1.6) results from repeated composition of (1.2) with itself; similarly for (1.7) and (1.3).

5. In addition to the examples memtioned in the Introduction, the case

$$\Phi(x) = \frac{1 - z}{1 - ze^x} \qquad (z \neq 1) \tag{5.1}$$

is of some interest. The resulting coefficients and polynomials were introduced by Euler [2, pages 487-491]; see also [3]. Let us denote the polynomials by $\eta_m(t, z)$, so that (2.4) becomes

$$\sum_{m=o}^{\infty} \frac{x^m}{m!} \eta_m(t, z) = \frac{(1 - z)e^{xt}}{1 - ze^x} \tag{5.2}$$

Now by (5.1) we have

$$\frac{\Phi(\lambda^{-1}x)}{\Phi(x)} = \frac{1 - ze^{x}}{1 - ze^{\lambda^{-1}x}}. \tag{5.3}$$

If $z = \zeta$, an γ-th root of unity, and we put $\lambda = k$, where $k \equiv 1 \pmod{\gamma}$, then the right member of (5.3)

$$= 1 + \zeta e^{x/k} + \cdots + \zeta^{k-1}e^{(k-1)x/k}.$$

(note that $1 + \zeta + \cdots + \zeta^{k-1} = 1$.) Comparison with (3.2) and Theorem 1 now yields

Theorem 4. *The polynomials $f_m(t) = \eta_m(t,\zeta)$ defined by* (5.1) *or* (5.2) *with $z = \zeta$, an γ-th root of* 1, *satisfy*

$$\sum_{r=0}^{k-1} \zeta^r \eta_m\left(t + \frac{r}{k}, \zeta\right) = k^{-m}\eta_m(kt,\zeta), \tag{5.4}$$

provided $k \equiv 1 \pmod{\gamma}$.

For $\zeta = -1$, (5.4) reduces to (1.3). By composition it is evident how (1.7) can be generalized.

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INTEGRALS AND EQUAL DIVISION SUMS

Morris Morduchow

Foreword

For f a continuous function on $[0,\ 2\pi]$ it is well known that

$$\lim_{m \to \infty} \sum_{j=1}^{m} f\left(\frac{2\pi}{m}j\right) \frac{2\pi}{m} = \int_0^{2\pi} f(x)dx.$$

Under what conditions on f does it follow that there is an integer m such that

$$\sum_{j=1}^{m} f\left(\frac{2\pi}{m}j\right) \frac{2\pi}{m} = \int_0^{2\pi} f(x)dx? \tag{1}$$

Similar questions have been studied recently by Szasz and Todd[1], who treated the limiting case in which $m \to \infty$, while the region of integration is infinite. Krishnan[2] has considered the case of a sufficiently large but finite m, and has derived conditions (involving the Fourier transform) under which eq. (1) holds when the integration is carried out over an infinite range.

It will be shown that the answer to the question raised in the present note can be expressed, and derived, in a particularly simple manner if the function is assumed to be representable by a complex Fourier series. It will be further shown that the familiar Euler-Maclaurin formula, which is pertinent here, can lead to a fallacious result if due caution is not exercised.

Derivation of Theorem

The following theorem answers the above question and its converse for a large class of functions.

Theorem. Let f be a function on $[0,\ 2\pi]$ whose complex Fourier series converges uniformly to it on $[0,\ 2\pi]$:

$$f(x) = \sum_{\nu=-\infty}^{\infty} A_\nu exp(i\nu x), \qquad 0 \le x \le 2\pi. \tag{2}$$

Then for this function equation (1) is valid, i.e.

$$\frac{1}{m} \sum_{j=1}^{m} f\left(\frac{2\pi}{m}j\right) = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx \tag{3}$$

if and only if

$$(4) \qquad \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} A_{\nu m} = 0$$

Proof. First note that

$$\sum_{j=1}^{m} exp(2\pi i\nu j/m) = \begin{cases} m \text{ if } \nu = km \\ 0 \text{ if } \nu\neq km,\ k=0,\ \pm 1,\ \pm 2,\ldots \end{cases}$$

Hence from (2)

$$\frac{1}{m}\sum_{j=1}^{m} f(\frac{2\pi}{m}j) = \frac{1}{m}\sum_{\nu=-\infty}^{\infty} A_\nu(\sum_{j=1}^{m} exp[2\pi i\nu j/m])$$

$$= \sum_{\nu=-\infty}^{\infty} A_{\nu m}.$$

But also from (2)

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = A_o.$$

Therefore equation (3) is valid if and only if condition (4) holds.

Equation (3), instead of (1), is used to emphasize the relation between the mean value of equally (finitely) spaced values of the function and the mean value of the function. Equation (4), of course, makes the problem of constructing particular functions which satisfy eq. (3) for a given integer m a very simple matter.

Examples

(a) Let f be a step function on $[0,2\pi]$ with m intervals of constancy each of length $2\pi/m$, and let f have left hand continuity at each point. Then eq. (1) or (3) obviously holds. Moreover, it is a fairly simple exercise to show that for this function the complex Fourier coefficients A_ν are such that $A_{\nu m} = 0$ for $\nu = \pm 1, \pm 2, \ldots$. Hence the condition (4) is satisfied by this function.

(b) For a positive integer p, eq. (3), with condition (4), leads directly to the result:

$$(5) \qquad \begin{cases} \displaystyle\sum_{j=1}^{m} \cos^p \frac{2\pi}{m} j = \frac{m}{2\pi}\int_0^{2\pi} \cos^p x\ dx \\ \qquad = 0 \text{ when } p \text{ is odd} \\ \qquad = m\,\dfrac{1\cdot 3\cdot 5\cdots(p-1)}{2\cdot 4\cdot 6\cdots(p)} \text{ when } p \text{ is even} \end{cases}$$

if and only if *none* of $|-p|$, $|2-p|$, $|4-p|, \cdots$, $|2p-p|$ are multiples of m other than zero. (This latter condition can be derived by observing that $\cos^p x = (1/2)^p[exp(-ipx)]\cdot[1+exp(2ix)]^p$ and expanding by the binomial theorem). The cases $p = 1$ and 2 have been applied in harmonic **analysis** (e.g., ref.3).

Euler-Maclaurin Formula

It is of particular interest to compare the theorem proven here with the implications of the familiar Euler-Maclaurin formula, since the latter is essentially an equation of the form (3) with the addition of correction terms. The Euler-Maclaurin summation formula, without the **remainder**, can be written in the form:

$$
(6) \quad \left\{ \begin{aligned}
&\sum_{j=0}^{m} f\left(\frac{2\pi}{m} j\right) = \frac{m}{2\pi} \int_0^{2\pi} f(x)dx + \frac{1}{2}\left[f(0) + f(2\pi)\right] \\
&\qquad + \frac{(2\pi/m)}{12}\left[f'(2\pi) - f'(0)\right] + \cdots \\
&\qquad + \frac{B_{2n}}{(2n)!}(2\pi/m)^{2n-1}\left[f^{[2n-1]}(2\pi) - f^{[2n-1]}(0)\right] + \cdots \quad (n \geq 1)
\end{aligned} \right.
$$

where the B_{2n} are Bernoulli numbers. It might at first sight appear from (6) that eq. (3) will be valid for any differentiable function $f(x)$ with a period of 2π, regardless of whether or not condition (4) is satisfied. That such, however, is not the case can be seen either by noting that the remainder term (e.g., ref. 4) for (6) after any number of terms will then not necessarily be identically zero*, or by referring to a derivation of (6). Eq. (6) can be derived[4] by assuming that $f(x)$ is expressible as a sum of exponential functions. Such a derivation involves an expansion of the function $(e^{\varphi} - 1)^{-1}$, where, in conjunction with eq. (2), φ would be essentially $(2\pi i\nu/m)$. Hence the cases of $e^{\varphi} = 1$, or $\nu = \pm km$ (k an integer), would have to be given special consideration, and this would be found to lead once again to a condition of the form (4). This illustrates the care with which infinite-series formulas, such as (6), must sometimes be applied, as well as the significance in such cases of the remainder term.

* If $f(x)$ is a polynomial, then the remainder does become identically zero after a finite number of terms, and this property can be used, for example, to obtain in a simple manner (cf. ref.4) a closed-form expression for the sum of the Pth (P a positive integer) powers of the first N integers.

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ON INTEGRATION OF FUNCTIONS OF THE FORM $e^{ax}f(x)$.

Reino W. Hakala

FOREWORD.- A process of induction due to DeMorgan is used to derive a formula by which functions $e^{-ax}f(x)$, where $f(x)$ is capable of successive differentiation, can be readily integrated. The integration process is unusual in that it merely involves finding successive derivatives of $f(x)$, & does not depend in any way on any prior knowledge or differentiation of the final result.

The usefulness of the formula is illustrated by a few examples: a simple proof of the whole-number values of the Γ function, simple derivations of the asymptotic expansion of the exponential-integral function, of the value of another interesting integral, & of more useful expressions for certain integrals than are now tabulated for them. Two new expressions for Π, also given here, arose accidentally from such investigations. Some formulas for certain Γ functions, related to these expressions for Π, are derived by induction. The general function $\int e^{+ax^b}x^n dx$, where $n + 1 = b, 2b, 3b, \ldots$, is also evaluated.

INTRODUCTION.- In 1842, Augustus DeMorgan gave a very interesting derivation of the value of the indefinite integral $\int e^x P dx$ ("The Differential & Integral Calculus". London: Baldwin & Cradock. P. 167). He proceeded by induction:

$$\frac{d}{dx}(e^xP) = e^xP + e^x\frac{dP}{dx} = e^x\left(1 + \frac{d}{dx}\right)P$$

$$\frac{d^2}{dx}(e^xP) = e^xP + 2e^x\frac{dP}{dx} + e^x\frac{d^2P}{dx^2} = e^x\left(1 + \frac{d}{dx}\right)^2P$$

$$\frac{d^3}{dx}(e^xP) = e^xP + 3e^x\frac{dP}{dx} + 3e^x\frac{d^2P}{dx^2} + e^x\frac{d^3P}{dx^3} = e^x\left(1 + \frac{d}{dx}\right)^3P \qquad .)^3P$$

and so on, whence

$$\frac{d^n}{dx^n}(e^xP) = e^x\left(1 + \frac{d}{dx}\right)^nP.$$

Then, since $\frac{d^{-1}}{dx^{-1}}u \equiv \int u\,dx$ (neglecting the constant of integration),

$$\int e^x P\,dx \equiv \frac{d^{-1}}{dx^{-1}}(e^x P) = e^x\left(1 + \frac{d}{dx}\right)^{-1} P,$$

and therefore

$$\int e^x P\,dx = e^x\left(1 - \frac{d}{dx} + \frac{d^2}{dx^2} - \dots\right)P,$$

by the binomial expansion. The result is more conveniently written

$$\int e^x f(x)\,dx = e^x(1 - D + D^2 - \dots)f(x)$$

(plus a constant), and is readily verified thru integration by parts.

GENERALIZATION.- By applying DeMorgan's method to the more general function $e^{\pm ax}f(x)$, it can be easily shown that

$$\int e^{\pm ax}f(x)dx = \frac{e^{\pm ax}}{a}\left(\pm 1 - \frac{D}{a} + \frac{D^2}{a^2} - \frac{D^3}{a^3} \pm \dots\right) f(x) \text{(plus const.)}$$

$$= - e^{\pm ax}\sum_0^\infty\left(\frac{\mp 1}{a}\right)^{p+1} D^p f(x) \text{(plus const.)} \qquad (1),$$

where the $\pm$'s are taken either all + or all -, and $\mp$ is the opposite sign. This formula includes DeMorgan's as a special case and is very useful as will be illustrated below.

APPLICATIONS.-

I. PEIRCE, NO. 403.- The formula given is

$$\int e^{ax}\, x^n\, dx = \frac{e^{ax}x^n}{a} - \frac{n}{a}\int e^{ax}\, x^{n-1}\, dx.$$

Making the appropriate substitution in (1), and keeping both signs, we find immediately that

$$\int e^{\pm ax}\, x^n\, dx = -\frac{e^{\pm ax}}{a}\left[\mp x^n + \frac{n}{a}x^{n-1} \mp \frac{n(n-1)}{a^2}x^{n-2} + \dots + (\mp 1)^{n+1}\frac{n!}{a^n}\right] + C$$

This result can be abbreviated

$$\int e^{\pm ax}\, x^n\, dx = -\frac{e^{\pm ax}}{a}\sum_0^n (\mp 1)^{p+1}\, p!\binom{n}{p}\frac{x^{n-p}}{a^p} + C,$$

which is both more general and more convenient to use than Peirce 403.

II. THE Γ FUNCTION. - If only the minus sign is taken in the above general formula, and the limits of integration 0 and ∞ are applied, it is found that

$$\int_0^{\infty} e^{-ax}\, x^n\, dx = 0\ (-\infty) + \frac{n!}{a^{n+1}} \text{ for } n = 0, 1, 2, 3, \ldots,$$

and diverges for $n = -1, -2, -3, \ldots$. Upon repeatedly applying Cauchy's theorem,

$$\lim_{x \to b} [f(x)/F(x)] = \lim_{x \to b} [f'(x)/F'(x)],$$

the indeterminate form is seen to reduce ultimately to zero. Letting $a = 1$ then gives the special case of the Γ function.

III. THE EXPONENTIAL-INTEGRAL FUNCTION. - If we let $f(x) = x^{-1}$ and $a = 1$ in (1), and take $\pm$ as $-$, we can evaluate

$$Ei\ (u) \equiv \int_{\infty}^{-u} e^{-x}\, x^{-1}\, dx :$$

$$\int e^{-x}\, x^{-1}\, dx = -\frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2}{x^2} - \frac{3!}{x^3} + \ldots\right) + C$$

$$\int_{\infty}^{-u} e^{-x}\, x^{-1}\, dx = \frac{e^{u}}{u}\left(1 + \frac{1}{u} + \frac{2}{u^2} + \frac{3!}{u^3} + \ldots\right).$$

IV. (a) EXPRESSIONS FOR Π.- i) Peirce, No. 497, states that

$$\int_0^{\infty} e^{-ax}\, x^{-\frac{1}{2}}\, dx = \sqrt{\frac{\pi}{a}}$$

Evaluation of the function by (1) and factoring out $\frac{1}{x}$ yields

$$\int e^{-ax}\, x^{-\frac{1}{2}}\, dx = -\frac{e^{-ax}}{a\ x}\left(1 - \frac{1}{2}x^{-1} + \frac{1}{2}\frac{3}{2}x^{-2} - \frac{1}{2}\frac{3}{2}\frac{5}{2}x^{-3} + \ldots\right) + C,$$

whereupon

$$\int_0^{\infty} e^{-ax}\, x^{-\frac{1}{2}}\, dx = \lim_{x \to 0} \frac{e^{-ax}}{a\sqrt{x}}\left(1 - \frac{1}{2}x^{-1} + \frac{1}{2}\frac{3}{2}x^{-2} - \ldots\right)$$

and

$$\sqrt{\pi} = \lim_{x\to 0} \frac{e^{-ax}}{\sqrt{ax}} \left[1 + \sum_{1}^{\infty} (-1)^p \frac{(2p-1)!}{(p-1)!\, 2^{2p-1}} x^{-p}\right] .$$

This result is intriguing, since no limitation was placed on the values that a might assume (except that a > 0). It is interesting for the further reason that

$$\frac{(2p-1)!}{(p-1)!\, 2^{2p-1}} = \frac{\Gamma(p+\frac{1}{2})}{\sqrt{\pi}} = \frac{(-1)^p \sqrt{\pi}}{\Gamma(\frac{1}{2}-p)},$$

as will be shown in Part (b), i and iii.

ii) The last-mentioned relationship prompts us to write

$$\Pi = (-1)^p\, \Gamma(\tfrac{1}{2}+p)\, \Gamma(\tfrac{1}{2}-p);\ p = 1, 2, 3, \ldots .$$

IV. (b) FORMULAS FOR CERTAIN Γ FUNCTIONS.- i) Given the recurrence relation, $\Gamma(t+1) = t\Gamma(t)$, and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we find that

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi},\ \Gamma(\frac{5}{2}) = \frac{3}{2}\,\frac{1}{2}\sqrt{\pi},\ \Gamma(\frac{7}{2}) = \frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi},\ \ldots ,$$

$$\Gamma(\frac{n}{2}) = \frac{(n-2)\ \ldots\ 5\cdot 3\cdot 1\cdot \sqrt{\pi}}{2^{(n-1)/2}} = \frac{(n-2)!\ \sqrt{\pi}}{(\frac{n-3}{2})!\ 2^{n-2}}$$

where n = 3, 5, 7, ... Now let n = 2p + 1. Then

$$\Gamma(p+\frac{1}{2}) = \frac{(2p-1)!\ \sqrt{\pi}}{(p-1)!\ 2^{2p-1}},\ p = 1, 2, 3, \ldots .$$

It is important to note that this formula does not include $\Gamma(\frac{1}{2})$. The latter is given, however, as a limit of the more general expression

$$\Gamma(p+\frac{1}{2}) = \frac{\Gamma(2p)\ \sqrt{\pi}}{\Gamma(p)\ 2^{2p-1}}$$

for all $p \neq 0, -1, -2, -3, \ldots$: i.e.,

$$\Gamma(\frac{1}{2}) = \lim_{p\to 0} \frac{\Gamma(2p)\sqrt{\pi}}{\Gamma(p)\ 2^{2p-1}} = \sqrt{\pi}\ .$$

ii) In view of the above, it can now be stated that

$$\int_0^\infty e^{-x^2} x^{2n}\, dx = \frac{1}{2}\Gamma(n + \frac{1}{2}) .$$

(See Peirce, No. 494.)

Altho the formula for $\Gamma(n + \frac{1}{2})$ does not include $\Gamma(\frac{1}{2})$, letting $n = 0$ in this integral gives the correct result, whence the value given for the integral is quite general (except of course that $n \neq 0, -1, -2, -3, \ldots$).

iii) If $\Gamma(-\frac{1}{2}), \Gamma(-\frac{3}{2}), \ldots$, are also considered, it is a simple matter to show that

$$\Gamma(\frac{1}{2} \pm n) = (\pm 1)^n \left[\frac{1 \cdot 3 \cdot 5 \ldots (2n - 1)}{2^n}\right]^{\pm 1} \sqrt{\pi} =$$

$$(\pm 1)^n \left[\frac{(2n - 1)!}{(n - 1)!\, 2^{2n-1}}\right]^{\pm 1} \sqrt{\pi}\ ;\ n = 1, 2, 3, \ldots .$$

This does not include $\Gamma(\frac{1}{2})$ either, but only

$$\ldots \Gamma(-\frac{3}{2}), \Gamma(-\frac{1}{2}), \Gamma(\frac{3}{2}), \Gamma(\frac{5}{2}), \ldots .$$

V. PEIRCE, NOS. 414 & 415.-

$$\int e^{ax} \sin bx\, dx = e^{ax}\left[\frac{\sin bx}{a} - \frac{b\cos bx}{a^2} - \frac{b^2 \sin bx}{a^3} + \frac{b^3\cos bx}{a^4} + \ldots +\right]C$$

$$= e^{ax}\left[\frac{a\sin bx - b\cos bx}{a^2}\right](1-(\frac{b}{a})^2+(\frac{b}{a})^4-(\frac{b}{a})^6 + \ldots +)C$$

$$= e^{ax}\,(a\sin bx - b\cos bx) / (a^2 + b^2) + C, \text{ where } a > b$$

is required for convergence of the infinite series. This is the same equation as in Peirce's Tables, except that Peirce does not give the necessary limitation $a > b$. The corresponding function for $\cos bx$ (No. 415) is readily evaluated the same way; here $a > b$ also.

VI. PEIRCE, NOS. 419-425.- The application of formula (1) to these functions is quite straightforward (as it always is), but working out a brief result for each of them (as in I & V) is a very long and tedious process, and is therefore left as an exercise for interested readers with much more patience and spare time than the author.

Another worthwhile, but even more tedious, exercise is to derive a formula for

$$\int e^{\pm ax} \sin^m px \cos^n qx \, dx \, .$$

To make it even more worthwhile (and tedious!), let m and n be any combination of +, -, and zero.

VII. $\int e^{ax^b} f(x)dx$.- This formulation includes many interesting functions, but DeMorgan's process of induction is unable to handle it. However, it is possible to treat indirectly the important class of functions covered by

$$\int e^{\pm ax^b} x^n \, dx \, ,$$

with the limitation that $n + 1 = b, 2b, 3b, \ldots$.

Apply formula (1) to

$$\int e^{\pm ay} y^k \, dy \quad (\text{cf. } \S\ 1) \, ,$$

taking $k = (n - b + 1)/b$. (The limitation previously mentioned arises because $k = 0, 1, 2, 3, \ldots$.) Multiply both sides of the result by $\frac{1}{b}$. The desired integral is then found by letting $y = x^b$.

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A DECISION METHOD FOR TRIGONOMETRIC IDENTITIES

Eliot Chamberlin and James Wolf

Occasionally one must investigate a trigonometric equation to see whether or not it holds identically, and identity proving is an important industry in trigonometry texts and courses. Failure to prove a proposed identity may result from the fact that it is not an identity or from insufficient ingenuity. In this note we give a mechanical procedure for deciding whether certain types of trigonometric equations, including the usual identities of trigonometry texts, are identities. Application of the mechanical procedure will not necessarily be the shortest or most instructive method of treating a given identity. Examples are given at the end.

1. TRIGONOMETRIC POLYNOMIALS. If $P(x)$ is a polynomial in $\sin a_i x$, $\cos a_i x$ $(i = 1,\dots,n)$ where the numbers a_i are rational, the following mechanical procedure reduces $P(x)$ to the form $q_1(\sin x/p)\cos x/p + q_2(\sin x/p)$ for some number p and polynomials q_1 and q_2. Write $a_i = m_i/p$ where p is the least common denominator of the numbers a_i and m_i is an integer, use the multiple angle formulas (which can be produced mechanically) to write P as a polynomial in $\sin x/p$ and $\cos x/p$, and replace $\cos^{2r} x/p$ by $(1 - \sin^2 x/p)^r$ and $\cos^{2r+1} x/p$ by $(1 - \sin^2 x/p)^r \cos x/p$ to obtain the above form.

THEOREM 1. If $q_1(v)$ and $q_2(v)$ are polynomials, $q_1(\sin u)\cos u + q_2(\sin u) \equiv 0$ if and only if all coefficients of q_1 and q_2 are zero.

For if we have an identity, $q_1^2(\sin u)\,(1 - \sin^2 u) \equiv q_2^2(\sin u)$; hence $q_1^2(v)\,(1 - v^2) \equiv q_2^2(v)$. But this is impossible unless q_1 and q_2 are identically zero since the left hand side has an odd number of prime factors $(1 - v)$ while the right hand side has an even number.

If $P(x_1,\dots,x_n)$ is a polynomial in $\sin a_i x_1$, $\cos a_i x_1,\dots,$ $\sin c_j x_n$, $\cos c_j x_n$ where the numbers $a_i,\dots,c_j$ are all rational, the above process applied to each variable in turn reduces P to a polynomial Q in $\sin x_1/p_1$, $\cos x_1/p_1,\dots,$ $\sin x_n/p_n$ where cosines

occur to no higher power than the first. Repeated application of theorem 1 then shows that $P \equiv 0$ if and only if every coefficient of Q is zero.

2. *IRRATIONAL MULTIPLIERS IN THE ARGUMENT*. A set of real numbers $b_1,\dots,b_k$ will be called *DEPENDENT* if $r_1b_1+\dots+r_kb_k = 0$ for some set of integers $r_1,\dots,r_k$ not all zero, otherwise $b_1,\dots,b_k$ are *INDEPENDENT*.

THEOREM 2. If $R(x)$ is a polynomial in $\sin b_1x$, $\cos b_1x,\dots,$ $\sin b_kx$, $\cos b_kx$ where $b_1,\dots,b_k$ are independent, let $S(v_1,\dots,v_k)$ be the result of replacing $b_1x,\dots,b_kx$ in R by $v_1,\dots,v_k$ respectively. Then $R(x) \equiv 0$ if and only if $S(v_1,\dots,v_k) \equiv 0$.

Clearly $R(x) \equiv 0$ if $S(v_1,\dots,v_k) \equiv 0$. Suppose $R(x) \equiv 0$; then by Kronecker's theorem (see reference [1], theorem 444) for any given $v_1,\dots,v_k$, $|b_ix/2\pi - p_i - v_i/2\pi|$ $(i = 1,\dots,k)$ can simultaneously be made as small as we please by proper choice of x and integers p_i, i.e. $\sin b_ix$, $\cos b_ix$ simultaneously take values close to $\sin v_i$, $\cos v_i$ respectively $(i = 1,\dots,k)$. But $R(x) \equiv 0$, hence $S(v_1,\dots,v_k) \equiv 0$ by continuity.

If $b_1,\dots,b_k$ of theorem 2 are not independent, there is an independent subset and all b_i are linear combinations with rational coefficients of the numbers of this subset; use of the addition formulas for sine and cosine then gives a function to which theorem 2 applies.

The preceding results give a mechanical reduction of the problem of proving trigonometric identities to the problem of determining whether some real constants (the coefficients) are zero and whether some real constants (the multipliers in the arguments) are independent. For rational numbers, this is mechanical.

Equations involving rational functions of the trigonometric functions are often called identities if the two sides of the equation are defined and equal except for isolated values of the variable. Such identities are easily treated by eliminating all trigonometric functions except sine and cosine, and multiplying through by the denominators; these vanish only at isolated points if they are not identically zero, and this possibility can be tested by the above methods.

EXAMPLE 1. Is the following equation an identity (in the above sense)?

$$(1)\quad \frac{\cos x/2 \sin x/6}{2\cos x/3 - 1} = \frac{\cos x/6 \sin x/2}{2\cos x/3 + 1}$$

It is obvious (or use theorem 1 with $u = x/3$) that neither denominator vanishes identically, hence (1) is an identity if and only if

$$(2)\quad \cos x/2 \sin x/6 \,(2\cos x/3 + 1) - \cos x/6 \sin s/2 \,(\cos x/3 - 1) = 0.$$

Using $\cos 2z = \cos^2 z - \sin^2 z$, $\sin 2z = 2\sin z \cos z$, $\cos 3z = \cos(2z+z) = \cos^3 z - 3\cos z \sin^2 z$, and $\sin 3z = 3\cos^2 z \sin z - \sin^3 z$, with $z = x/6$, the left hand side of (2) reduces to $C^5(-4S) + C^3(4S) + C(4S^5 - 4S^3)$ where $C = \cos x/6$ and $S = \sin x/6$. Reduction to a polynomial linear in C by use of $C^2 = 1 - S^2$ gives the zero polynomial, hence (1) is an identity.

EXAMPLE 2. *Is the* following equation an identity?

$$(1)\quad \sin 2\pi x \cos^2 x + \sin 2x \cos^3 \pi x = 0$$

The multipliers in the argument are 1, 2, π, 2π, and these are rationally dependent on 1 and π which are independent (the lroof of this is of course not contained in the above decision method). Then (1) is an identity if and only if

$$(2)\quad \sin 2y \cos^2 x + \sin 2x \cos^3 y = 0.$$

Using the method of paragraph 1 first with respect to x, (2) holds if and only if $\cos x\,(2\sin x \cos^3 y) + (\sin 2y - \sin^2 x \sin 2y) = 0$, and according to theorem 1, this holds only if $q_1 = 2\sin x \cos^3 y$ is identically zero. Since the coefficient $2\cos^3 y$ is not identically zero in y, (1) is not an identity which is also obvious, in this particular case, from setting $x = 1$ in (1).

[1] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, Oxford, 1938

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CURRENT PAPERS AND BOOKS

(Ctd. from inside back cover)

TECHNOLOGICAL APPLICATIONS OF STATISTICS. By L. H. C. Tippett, John Wiley and Sons, Inc., New York, 1950, IX + 189 pages. $3.50.

When a strong American center of learning invites a distinguished European scholar to deliver a series of lectures, the result is likely to prove gratifying, andwhen the lectures, somewhat amplified, appear in book form, the result is likely to represent a contribution to the subject. This is the situation with respect to Tippett's *TECHNOLOGICAL APPLICATIONS OF STATISTICS*. Mr. L. H. C. Tippett, head of the Mechanical Processing Division of the British Cotton Industry Research Association, was invited to give a course of lectures on the above subject at the Massachusetts Institute of Technology, and the book is the "write up" of these lectures. The audience for the lectures, we are told, consisced of industrialists, students, and practicing statisticians. The author hopes to reach a wider audience with his book. Students study descriptive and experimental physics before undertaking mathematical physics. Our author thinks that, similarly, a student should first acquire a "feel" for statistics through examples treated arithmetically. "The mathematics", he says, "can follow later." "The book can be regarded as an introduction and companion to a systematic textbook in applied statistics." Thus we understand why the author refers in a general way "to the textbooks" for mathematical proofs. His use of the expression "it can be shown" leaves an unsatisfied feeling with a reader accustomed to mathematical thinking and should stimulate him to further study.

For a textbook, a teacher thinks of decimally numbered chapters and articles with these decimal labels at the top of each page and of copious lists of exercises provided with answers for the odd-numbered ones. These are devises that aid in the teaching process, and these the teacher does not find in Mr. Tippett's book, but he is compensated by unusually clear exposition accomplished by many carefully worked examples. Thus there is no lack of motivation as sometimes happens when the student is given much theory without his seeing its usefulness to him in solving his problems. But the author warns that his "arithmetic method" can only be really

(Continued on page 89)

CATENARY AND TRACTRIX IN NON-EUCLIDEAN GEOMETRY

Curtis M. Fulton

In this paper we give definitions of the catenary and the tractrix in Hyperbolic Geometry, which bear a close analogy to the corresponding Euclidean definitions. We then show that the well known relation between the two curves is preserved.

We use for our purpose limiting surface coordinates x, y, z. In the Hyperbolic Space we consider a triply orthogonal system of surfaces which intersect in coordinate curves that are mutually perpendicular. Specifically, x = const. is a family of limiting surfaces, y = const. and z = const. represent planes. The coordinate curves are limiting curves and straight lines, the latter being parallel. Let P(x, y, z) and P_1 (ξ, η, ζ) be any two points and cos , cos β, cos γ the direction cosines of the directed line segment PP_1 with respect to the coordinate curves through P. Then the following relations will hold [2]:

$$\xi = x - \log\left(\cosh PP_1 - \sinh PP_1 \cos \;\right); \tag{1}$$

$$\eta = y + \frac{e^x \sinh PP_1 \cos\beta}{\cosh PP_1 - \sinh PP_1 \cos\alpha},$$

$$\zeta = z + \frac{e^x \sinh PP_1 \cos\gamma}{\cosh PP_1 - \sinh PP_1 \cos\alpha}. \tag{2}$$

These formulas show quite naturally the metric significance of our coordinates. Setting $\cos a_{\alpha} = 1$ in (1), we find $\xi = x + PP_1$, which means that x measures distances between the limiting surfaces along the parallel lines. If in (2) $\cos\alpha = 0$ and $\cos\beta = 1$, we have $\eta = y + e^x \tanh PP_1$. Thus, [5, p. 139], $e^{-x}(\eta - y)$ is the arc of a limiting curve intercepted by two parallels in a plane z = constant. As an immediate consequence we record the equations for a translation of axes to the new origin (h, k, l),

$$\bar{x} = x - h,\quad \bar{y} = e^{-h}(y - k),\quad \bar{z} = e^{-h}(z - l). \tag{3}$$

Let x = x(s), y = y(s), z = z(s) be the parametric equations of a curve with the arc length s as parameter. The direction cosines of the tangent are given by [2]

$$\cos\alpha_T = x',\quad \cos\beta_T = e^{-x}y',\quad \cos\gamma_T = e^{-x}z', \tag{4}$$

the accents indicating derivatives with respect to s. These direction cosines satisfy the equation

(5) $$x'^2 + e^{-2x}(y'^2 + z'^2) = 1.$$

As a special case we obtain from (1) and (2) the parametric equations of a straight line, writing σ for the arc length PP_1. The corresponding direction cosines (4) will be

$$\frac{d\xi}{d\sigma} = -\frac{\sinh\sigma - \cosh\sigma\cos\alpha}{\cosh\sigma - \sinh\sigma\cos\alpha},$$

(6) $$e^{-\xi}\frac{d\eta}{d\sigma} = \frac{\cos\beta}{\cosh\sigma - \sinh\sigma\cos\alpha},$$

$$e^{-\xi}\frac{d\zeta}{d\sigma} = \frac{\cos\gamma}{\cosh\sigma - \sinh\sigma\cos\alpha}.$$

We now consider a perfectly flexible string of constant density equal to unity. Let $T = T(s) > 0$ denote the magnitude of the tension, which is a tangential force. Let $F > 0$ be the magnitude of the external force per unit mass and designate its direction cosines by $\cos\alpha_F$, $\cos\beta_F$, $\cos\gamma_F$. If there is no external force, the string hangs in a straight line; T is constant and the direction of the tension is determined by (6). If this is not the case, the first component of the tension undergoes a change along the element of arc Δs equivalent to

$$T(s + \Delta s)\,x'\,(s + \Delta s) + T(s)\,\frac{\sinh\Delta s - x'\cosh\Delta s}{\cosh\Delta s - x'\sinh\Delta s}.$$

Here the subtrahend represents just what would be the tension, if no external force were present. Adding $\Delta s F\cos\alpha_F$ to the above expression we must obtain zero, since the element Δs is in equilibrium. The same is true for the remaining components. As $\Delta s \to 0$, we are led to the fundamental equations of equilibrium, which, partly with the aid of (5), can be written

$$T'x' + T\,[x'' + e^{-2x}(y'^2 + z'^2)] + F\cos\ _F = 0,$$

(7) $$Te^{-x}(y'' - 2x'y') + T'e^{-x}y' + F\cos\beta_F = 0,$$

$$Te^{-x}(z'' - 2x'z') + T'e^{-x}z' + F\cos\gamma_F = 0.$$

Equations (7) easily yield expressions for the curvature $\frac{1}{R}$, if we take the direction cosines with the opposite signs and define $F = \frac{1}{R}$ for $T = 1$. Cf. [2]. Indicating the direction of curvature

by means of the subscript N and using (4), equations (7) may be rewritten as

$$\frac{T}{R}\cos\alpha_N + T'\cos\alpha_T + F\cos\alpha_F = 0,$$
$$\frac{T}{R}\cos\beta_N + T'\cos\beta_T + F\cos\beta_F = 0,$$
$$\frac{T}{R}\cos\gamma_N + T'\cos\gamma_T + F\cos\gamma_F = 0.$$

We now assume that the string is acted on by a force parallel to the direction of the x-coordinate, so that $\cos\alpha_F = 1$ and $\cos\beta_F = \cos\gamma_F = 0$. As we readily observe on account of the last two equations (7), $Te^{-2x}y'$ and $Te^{-2x}z'$ will then be equal to arbitrary constants. It follows that y and z satisfy a linear equation, which represents a plane cutting the limiting surfaces x = const. orthogonally. Without loss of generality we may take this plane as z = 0, in which x and y are limiting curve coordinates [5, p. 165]. Thus, equations (7) reduce to

$$\frac{d}{ds}(Tx') + Te^{-2x}y'^2 = -F,$$
$$Te^{-2x}y' = \frac{1}{C_1}.$$

Here we have written $\frac{1}{C_1}$ for the arbitrary constant: $y' = 0$ would correspond to the trivial case of a straight line. Furthermore we may assume $\frac{1}{C_1} > 0$, thus choosing a positive sense on the curve. Combining the two equations we can eliminate T and have

$$\frac{1}{C_1}y'\left[K\frac{d}{dy}\left(e^{2x}\frac{dx}{dy}\right) + 1\right] = -F. \tag{8}$$

We propose to consider two cases, the first case being that of a suspension bridge. If the acceleration of gravity acts in the direction of the x-coordinate, its magnitude will be ge^{2x} with a constant $g > 0$. Cf. [2] and [3, p. 131]. Assuming that the cable of the suspension bridge supports a roadbed of uniform linear density on a level x = constant, F will be proportional to y', i.e. $F = \rho y'$, $\rho > 0$. Integrating (8) we obtain

$$e^{2x} = -(1 + \rho C_1)(y - C_2)^2 + C_3,$$

where C_2 and $C_3 > 0$ are arbitrary constants. Such an equation represents a parabola: the locus of points equidistant from a fixed point and a fixed limiting curve.

For our second case we suppose that the string or chain hangs under its own weight. Then $F = ge^{2x}$ and making use of (5) we write (8) in the form

$$\frac{\dfrac{d^2}{dy^2}\left(\dfrac{1}{2}e^{2x}\right)+1}{\sqrt{e^{2x}+\left[\dfrac{d}{dy}\left(\dfrac{1}{2}e^{2x}\right)\right]^2}} = -gC_1.$$

Solving this differential equation by standard methods we find

$$\text{(9)} \qquad \cosh gC_1\,(y - C_3) = \frac{gC_1C_2 + 1 - \frac{1}{2}g^2C_1{}^2e^{2x}}{\sqrt{2gC_1C_2 + 1}},$$

$C_2 > 0$ and C_3 being arbitrary constants. Letting

$$\operatorname{csch} a = \sqrt{2gC_1C_2}\ , \qquad e^{-h} = \sqrt{\frac{gC_1}{2C_2}}\ , \quad k = C_3\ ,$$

equation (9) can be written in another form, namely,

$$\cosh \frac{e^{-h}(y - k)}{\sinh a} = \frac{\cosh^2 a + \sinh^2 a - e^{-2h}e^{2x}}{2\cosh a \sinh a}\ .$$

Taking (3) into account, we eventually reduce it to

$$\cosh \frac{\bar{y}}{\sinh a} = \frac{\cosh^2 a + \sinh^2 a - e^{2\bar{x}}}{2\cosh a \sinh a}$$

the equation of a catenary, where the geometrical significance of the positive constant a will appear later. It may also be shown that $T = \frac{g}{2}(e^{2h} - e^{2x})$, a relation whose mechanical meaning is discussed in [4, p. 322].

We now define the tractrix as the plane curve for which the length of the tangent intercepted between the point of contact and the limiting curve $x = 0$ is always equal to a. We can express this condition by means of (1), taking $\xi = 0$, $PP_1 = a$, and $\cos \alpha = x'$ according to (4). Thus,

$$x = \log(\cosh a - x'\sinh a)\ ,$$

and, using (5) for y' we have at once

$$\frac{dy}{dx} = \frac{e^x\sqrt{\sinh^2 a - (\cosh a - e^x)^2}}{\cosh a - e^x}\ ,$$

where we restrict ourselves to $x' > 0$, $y' > 0$. The integral of this equation is found to be

$$(11)\qquad y = -\sqrt{\sinh^2 a - (\cosh a - e^x)^2} + \sinh a \log \frac{\sinh a + \sqrt{\sinh^2 a - (\cosh a - e^x)^2}}{\cosh a - e^x} + C.$$

Let us set $C = 0$ in the preceding equation, thus effecting a translation (3), so that the tractrix will pass through $(-a, 0)$. It is convenient to introduce a positive parameter t equal to the logarithm in (11). The resulting parametric equations are

$$(12)\qquad \begin{aligned} x &= \log(\cosh a - \sinh a \operatorname{sech} t), \\ y &= -\sinh a \tanh t + t \sinh a. \end{aligned}$$

We could easily show, using (2) that the intercept of the tangent of the tractrix on $x = 0$ equals $t \sinh a$.

In order to find parametric equations of the evolute [1, p. 192] of a given plane curve we return to (4). The direction cosines of the normal, with the proper signs for our case, are

$$(13)\qquad \cos\alpha_N = -e^{-x}y', \quad \cos\beta_N = x'.$$

In this way we find for the curvature $\frac{1}{R}$, which was defined in connection with (7)

$$(14)\qquad R = \frac{-e^{-x}y'}{x'' + e^{-2x}y'^2} = \frac{x'}{e^{-x}(y'' - 2x'y')}.$$

Applying (14) to a circle of radius r, its curvature turns out to be $\operatorname{ctnh} r$ [1, p. 134]. We therefore define the radius of curvature r of any curve by means of

$$(15)\qquad \tanh r = R,$$

which becomes meaningless when $R > 1$. Carrying over the Euclidean definition of the evolute, (1) and (2) will give its parametric equations

$$(16)\qquad \begin{aligned} \xi &= x - \log(\cosh r - \sinh r \cos\alpha_N), \\ \eta &= y + \frac{e^x \sinh r \cos\beta_N}{\cosh r - \sinh r \cos\alpha_N}, \end{aligned}$$

where the proper substitutions from (13), (14), (15) have to be made.

To determine the evolute of the tractrix (12) it is not necessary to actually introduce the arc length. By a simple manipulation we obtain from (12) making adequate use of (5)

$$x' = \operatorname{sech} t, \quad e^{-x}y' = \tanh t,$$

and, using (14)

$$R = \frac{\sinh t}{\operatorname{ctnh} a - \cosh t} .$$

With these relations, (16) can be written

$$e^{2\xi} = \cosh^2 a + \sinh^2 a - 2\cosh a \sinh a \cosh t ,$$
$$\eta = t \sinh a .$$

Now, elimination of t leads to (10), the equation of a catenary.

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University of California, Davis

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, as a teacher, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

ALLIGATION - IT'S MEANING AND USE

E. Justin Hills

Alligation, often called the Rule of Mixture, is a method of compounding ingredients. Alligation (Latin - alligare, to bind to, to tie up) teaches how to mix ingredients according to any intent or design. Thus it can be used to solve quickly any mixture problem. It also has wide possible application in pharmacy, merchandising, and other fields.

There are two types of alligation - medial and alternate. Alligation medial is nothing more than finding the mean value of a series of values. Alligation alternate is very important since it shows how much of various kinds of amounts may be taken to make up any assigned quantity of a compound. This article emphasizes the importance of getting familiar with alligation alternate.

Let us consider a simple alligation alternate problem. How much 5% solution and 20% solution must be combined to make a 14% solution? By simultaneous equations, we would have:

$$\begin{array}{lll} 5\%x + 20\%y = 14\%A & \text{So } x = 2/5 \text{ of } A & \text{And } x:y = 2:3 \\ \;\;\;\; x + \;\;\;\;\;\; y = \;\;\;\;\; A & \;\;\;\;\;\;\; y = 3/5 \text{ of } A & \end{array}$$

There are three alligation alternate procedures that can be carried out. The first one is several hundred years old. It is in Edward Hatton's "An Entire System of Arithmetic", which was published in about 1732. Here it is:

Per cents		Differences	
5		6	Therefore 6/15 of desired solution is 5% solution, and
	14		9/15 of desired solution is
20		9	20% solution.
		15	

That is, 2:3.

The second alligation alternate procedure uses plus and minus signs. The second has been used for many years in retailing when a retailer is solving certain averaging markup problems.

Desired per cent	14%	14%
Given per cents	5%	20%
	+9%	-6%
	x 2	x 3
	+18%	-18%

First take the differences of the desired and given per cents, noting the signs. Then determine the desired multipliers needed to make the plus product equal the minus product. These products give the 2:3 ratio that has been noted earlier.

The third alligation alternate procedure uses a method called the 'link method'. Several books printed during the 19th century referred to this method. It is used by many pharmacists, and as late as 1938 it was used in a McGraw-Hill book entitled 'Pharmaceutical Mathematics', by Edward Spease. The following diagram, showing one link, is used:

$$\begin{array}{ccc} & 14 & \\ 5 & & 20 \\ \hline 6 & & 9 \\ & 2:3 & \end{array}$$

Put the 14 between the 5 and the 20 and slightly above. Disregarding signs, take the differences and put them opposite to what is expected. Since 6:9 equals 2:3, the same ratio appears again. Since 2 + 3 = 5, 2/5 of the 14% solution is the 20% solution.

If we have 20 cc of 5% solution and desire to know how much 20% solution is needed to make the desired 14% solution, change 2:3 into 20:30. Thus 30 cc of 20% solution is needed.

If three or more amounts are to be combined to make a desired compound, the link method is the only satisfactory solution. Such a problem cannot be solved directly by simultaneous equations since it would have only two equations with three or more unknowns. That is, there will be more than one possible solution to such a problem.

Let us consider a problem with three amounts. How much is needed of a 5% solution, a 10% solution, and a 20% solution to make a 14% solution?

First procedure - the oldest one.

Per cents		Differences
5		6
10		6
	14	
20		9 + 4
		25

Each value less than 14 has the difference between 20 and 14 next to it. The only value greater than 14 has the differences between both per cents and 14% next to it.

Unfortunately this method gives only one solution.

Second procedure - the one used in retailing.

Desired per cent	14%	14%	14%
Given per cents	5%	10%	20%
	+9%	+4%	-6%

It would be rather difficult to find the needed multipliers so that the sum of the plus values would equal the minus value. The 6:6:13 found in the first procedure will do it. Note that

+9%	+4%	-6%		
x 6	x 6	x 13		
+54%	+24%	- 78%	=	0

Many more multipliers can be found when the link method is used.

Third procedure - the link method.

	14	
5	10	20
	6	4
	3:2	
6		9
	2:3	

5	10	20
	3	2
2		3
2	3	5
	2:3:5	

Since there are three amounts, two links are needed. Rewrite the given amounts and put the parts of each ratio under the proper amount. Under 20, the 2 and 3 must be combined to finally get 2:3:5. Since 2 + 3 + 5 = 10, 2/10 or 1/5 of the desired solution is 5% solution, 3/10 of the desired solution is 10% solution, and 5/10 or 1/2 of the desired solution is 20% solution.

Since 3:2 = 6:4 or 2:3 = 4:6, two other possible solutions would be:

5	10	20
	6	4
2		3
2	6	7
	2:6:7	

5	10	20
	3	2
4		6
4	3	8
	4:3:8	

In like manner many other combinations can be determined. To get the answer found in the first procedure, and used in the second procedure, merely change 3:2 to 6:4 and 2:3 to 6:9. Or use the values found originally before the ratio values were reduced. That is:

5	10	20
	6	4
6		9
6	6	13
	6:6:13	

If there are more than three amounts being combined, the first and third procedures can be carried out by at least four ways, and the second lrocedure would be used for check purposes only. We can have two links, three links, ir four links when using four amounts. We can have as many as seven links when using five amounts, or only three links.

This time let us match the first and third procedures using four amounts. Consider the following problem. How much is needed of a 5% solution, a 10% solution, a 15% solution, and a 20% solution to make a 14% solution? hote combinations a), b), c), and d).

a) By first procedure

```
Per cents    Differences
  5              6
 10              1
    14
 15              4
 20              9
     6:1:4:9
```

By third procedure

```
      14
5   10   15   20        5   10   15   20
     1    4                  1    4
       1:4              2              3
5              9        2    1    4    3
       2:3                 2:1:4:3
```

b) By first procedure

```
Per cents    Differences
  5              1
 10             1 + 6
    14
 15             4 + 9
 20              1
     1:7:13:1
```

By third procedure

```
      14
5   10   15   20        5   10   15   20
     1    4                  1    4
       1:4              1         9
1         9                  6         1
     1:9                1    7   13    1
6              1           1:7:13:1
       6:1
```

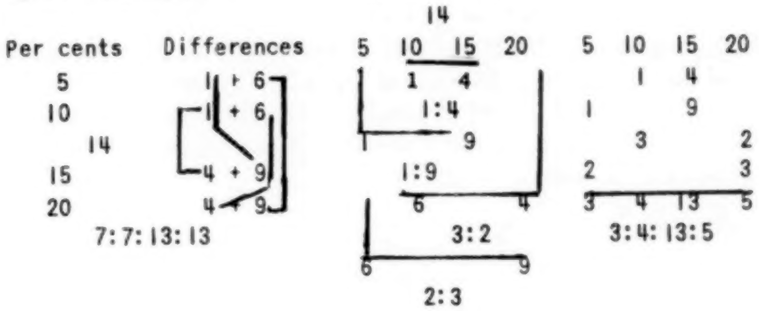
c) By first procedure

```
Per cents    Differences
  5              1
 10              6
    14
 15              9
 20              4
     1:6:9:4
```

By third procedure

```
      14
5   10   15   20        5   10   15   20
1         9             1         9
     1:9                     3         2
     6         4        1    3    9    2
       3:2                 1:3:9:2
```

d) By first procedure   By third procedure



Note that the links in the third procedure match perfectly the differences taken inthe first procedure. Also note that any difference used orany link drawn must involve two amounts one less and one greater than the desired figure.

It is the hope of this writer that you can make good use of these ideas and can find many applications. The writer has used this set of ideas in his classes and in some of his writing, especially in retailing. The response has been very interesting. None of these procedures are new, they have just been brought up to date.

Los Angeles City College

(Ctd. from page 78)

successful if the student works through carefully all of the numerical examples. "They are treated as problems in technology as well as in statistics." The data for these examples are from actually published research; no fictitious data are used. Nor is there the usual collection of statistical tables. Only a few specific references are given in the footnotes, but a general bibliography of twenty-one well-chosen and classified books and pamphlets is appended at the end of the book.

A recent writer remarks that quality control has become almost a cult in the industrial world. Mr. Tippett appropriately starts his book with this important subject. Part I of the book is entitled *THE ROUTINE CONTROL OF QUALITY* and includes about one-third of the total number of pages. It discusses control charts, including the control of the fraction defective, and acceptance sampling. The latter subject is especially well-treated in the longest chapter of *PART I* although the author tells us, "here we deal with only a few simple situations for the purpose of introducing the main ideas and quantities used in sampling inspection." *PART II, INVESTIGATION AND EXPERIMENTATION*, resembles a course in general statis ics with all applications, of course, in the industrial field. Some of the topics are: theory of small samples, analysis of variance, and correlation. Analysis of variance is treated to greater length than the other topics, and use is made of the almost universally accepted

tabular form for presenting the results of the computations. Simple and multiple correlation are discussed, but partial correlation is only mentioned. The theory of correlation is carried to the point of testing the signific nce of results by means of the variance ratio F. The final chapter deals with planning an investigation. The whole book demonstrates the very important place that statistics has attained in modern industry.

University of Arizona R. F. Graesser

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Ind.

*MATHEMATICAL THINKING AIDS IN INTELLIGENT CITIZENSHIP**

Alice Patricia Kenny, Albany High School, Albany, New York

Functions of angles and the Pythagoran Theorem have little use as such in everyday life. Students not planning mathematical or scientific careers often wonder why algebra and geometry are required for college entrance.

The study of numbers teaches one to set down ideas in logical order, and from specified facts to proceed in definite paths to a final conclusion. Perhaps the steps and reasons requisite to a geometric proof are best illustrative of this process. From the given material one extracts an idea leading in the direction of the desired result, and by means of rules and postulates builds up a solid superstructure, each component of which fits harmoniously and essentially into the whole.

The citizen who has studied mathematics will apply his understanding to the issues brought before him for his consideration. He will take the given facts, by basic rules develop what he learns, and build up a solid core of informed opinion. This core will be able to withstand attack and succeed in argument.

A habitually mathematical thinker keeps his thoughts in order. Not only is he able to make wise decisions on grave issues; he is a better adjusted person when his mind is clear of extraneous matter.

The human mind cannot be entirely governed by mathematics; consequently mathematics will not save the world. But minds conditioned to clear thinking can bring a saner atmosphere to national and international relations. It is thus easily conceivable that the study of mathematics will aid in providing tomorrow's democracy with wise leadership.

Submitted by Sister Noel Marie C.S.J.

**One of the essays submitted in a contest sponsored by the Mathematics Department of the College of St. Rose, Albany, New York*

THE CAT

I wonder if the cat on his ninth lap (eight lives expended) calculates while purring, "Eleven per cent is roughly what I've left: what's gone is eighty-eight-point-eight...recurring."

London Punch, page 293, Sept. 12, 1951

Submitted by W. R. Ransom Tufts College

ON TOPOLOGICAL REPRESENTATION OF GROUPS

In the March-April issue of this magazine D. Ellis proves that every countable group can be represented as a group of motions of a metric space. In view of this the following observations, while quite trivial, may be of some interest.

Let X be a topological space, let I be a set of indices, and let X^I be provided with the customary product topology. Then it is clear that any group G of permutations of I is isomorphic with a group of homeomorphisms of X^I onto itself, under the correspondence which assigns to every $g \in G$ the homeomorphism

$$(x_i \mid \quad i \in I) \qquad (x_{g(i)} \mid \quad i \in I)$$

In this way every group G can be represented as a group of homeomorphisms of X^G onto itself.

Taking X to be the discrete two-point space whose elements are 0 and 1, we deduce:

1. Every group is isomorphic with a group of homeomorphisms of a compact totally disconnected Hausdorff space onto itself.
2. Every countable infinite group is isomorphic with a group of homeomorphisms of the Cantor discontinuum onto itself

Taking X to be the closed unit interval, we deduce:

1. Every infinite group is isomorphic with a group of homeomorphisms of a compact connected Hausdorff space onto itself.
2. Every countable infinite group is isomorphic with a group of homeomorphisms of the Hilbert cube onto itself.

Burroughs Corporation Joel Pitcairn

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.

PROPOSALS

181. Proposed by C. W. Trigg, Los Angeles City College.

Each word or phrase in the following semi-coherent story is a mutation of the name of a mathematician. Identify:

(1) BREAD LOVER, (2) HIRES GAL, (3) GAVE, (4) CHEF LIST FOR, (5) MEAL, (6) MAKE RUM, (7) RAW GIN, (8) GREEN NOG, (9) GLEAN, (10) GUY'S HEN, (11) DRIED TO, (12) OUR FIRE, (13) SO IN POT, (14) RUIN A CLAM, (15) STIR OUT SAND, (16) NO PEA, (17) ELK IN, (18) AY! BOIL, (19) SOUP IN OLLA, (20) SKIN COD, (21) USC CORN PIE, (22) HER COB, (23) IS LEFT, (24) NO HAM, MAC, (25) COP CAKE, (26) FUMED CAFE, (27) HAL SET, (28) HUT TABLE, (29) ATE THE SUET, (30) ATE LOT, SIR, (31) SETS O. K., (32) NICE MODES, (33) SEE BULGE, (34) MAR BELT, (35) TABLE RIM, (36) INK MARK, (37) THEREON, (38) SHAME.

182. Proposed by E. P. Starke, Rutgers University.

a) Show that $f(a) = \sum_{j=1}^{\infty} [\frac{a}{j}] = \sum_{k=1}^{a} v(k)$

Where the brackets indicate the greatest integer function and $v(k)$ is the number of divisors of k.

b) Is there any limit to the ratio $f(a)/a$ as $a \to \infty$?

183. Proposed by D. L. MacKay, Manchester Depot, Vermont.

Construct a triangle ABC having given, in position, the circumcenter O, the foot D of the altitude from A and the point of intersection U of the bisector of angle A with the side BC.

184. Proposed by T. F. Mulcrone, Spring Hill College.

Show that in the sequence $1/n, 2/n, 3/n, \ldots, (n-1)/n$, where n is a positive integer greater than 2, an even number of the terms are fractions in lowest terms.

185. Proposed by Francis L. Miksa, Aurora, Illinois.

Four players, A, B, C, D, sit around a circular table. Each man has a number of matches in front of him. They play a match game as follows. First A removes enough matches from his pile and gives to his three other friends enough matches to multiply their holdings by 2, next B removes enough matches from his pile and gives to the three other players enough matches to multiply their holdings by factor 3, continuing C uses factor 4, D uses factor 5, A uses factor 6, and lastly B uses factor 7.

After those six plays it is found that each man has exactly the same amount of matches he started with at the begining of the game.

What is the smallest number of matches each man could have at the begining of the game? Develop a formula for N men, n plays, $n > N$, and n different factors.

186. Proposed by Edwin C. Gras, US Naval Academy.

Prove the following identity involving binomial coefficients:

$$\sum_{r=0}^{k} (-1)^r \; C_{k-1}^{n+r} \cdot C_r^k = 0.$$

SOLUTIONS

Late Solutions

159. Thomas F. Mulcrone, Spring Hill College.

155. John Link, Wisconsin High School, Madison, Wisconsin.

Integer Triples

161. [March 1953] Proposed by E. P. Starke, Rutgers University.

Show that there are infinitely many sets of three integers whose product equals the sum of their squares.

I. Solution by Harry M. Gehman, University of Buffalo. If (a, b, c) is a solution of $xyz = x^2 + y^2 + z^2$, then $(ab - c, a, b)$ is also a solution. Moreover if $a \geq b \geq c \geq 0$, and if $b \geq 2$, then $ab - c > a \geq b$, and the second solution is different from the first. Thus from the solution $(3, 3, 3)$, an infinite number of other solutions may be generated: $(6, 3, 3)$, $(15, 6, 3)$, $(87, 15, 6)$, etc..

II. Solution by Joseph Rosenbaum, Hartford, Connecticut. Observing that $(3, 3, 3)$ is a solution it is verified that if $(x, y, 3)$ is a solution, then $(x', y', 3)$ is also a solution where $x' = 21x - 8y$ and $y' = 8x - 3y$. The initial solution $(3, 3, 3)$ together with the transformation gives an infinite sequence of solutions.

Also solved by Thomas Griselle, C. W. Trigg, and the proposer.

The proposer pointed out that the corresponding problem with four integers admits the same treatment and yields the analogous result: all solutions of $x^2 + y^2 + z^2 + w^2 = kxyzw$ are obtained from $(2, 2, 2, 2)$ with $k = 1$ by use of the fact that whenever (x, y, z, w) is a solution so also is (x, y, z, w') where $w' = kxyz - w$.

For five or more letters, again an infinite sequence of solutions can be obtained from a given solution in the same manner, but it is no longer true that all solutions arise from one basic solution.

Some interest attaches to the problem of finding three integers whose sum of squares is a divisor of their product. For arbitrary k, there are solutions for

$$k(x^2 + y^2 + z^2) = xyz. \tag{5}$$

We need only take $x = kx'$, $y = ky'$, $z = kz'$, where $x'^2 + y'^2 + z'^2 = x'y'z'$ as determined above. These are not the only solutions. In fact, let x', y', z' be arbitrary integers and put $P = x'y'z'$, $S = x'^2 + y'^2 + z'^2$. Then $k = P$, $x = Sx'$, $y = Sy'$, $z = Sz'$ will satisfy (5).

Solutions in which x, y, z are relatively prime in pairs may be obtained by among others, the following procedure. Suppose k is a divisor of z (k must be a divisor of xyz) and put $z = ak$. Then (5) reduces to a quadratic Diophantine equation from which solutions are easy to find for each value of a. To obtain solutions in relatively prime integers we will need to note that a is a divisor of $x^2 + y^2$

and that $a - 2$ and $a + 2$ are divisors of $(ay - 2x)^2 + (2ak)^2$. However, no divisor of $u^2 + v^2$ can contain a prime factor of the form $4c - 1$ unless both u and v are divisible by the same factor. Hence, possible values of a are limited. Details are quite routine but rather tedious. Some examples are:

$$k = 12,\ x = 29,\ y = 929,\ z = 1752;$$
$$k = 18,\ x = 1°1,\ 6 = 485,\ z = 2628.$$

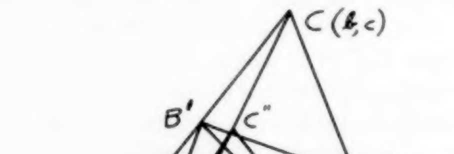
From any one solution an infinite sequence of solutions may be obtained by noting that, if (x, y, z) is a solution of (5), so also is (x, y', z) where $y + y' = xz/k$. Since $z = ak$, y' is an integer.

A VARIABLE TRIANGLE

162. [March 1953] Proposed by Howard Eves, State University of New York.

A variable triangle $A'B'C'$ of constant area is inscribed in a triangle ABC, so that A' lies on BC, B' on CA, and C' on AB. If AA', BB', CC' are divided by A'', B'', C'', respectively, in the same ratio, show that the triangle $A''B''C''$ has a constant area.

Solution by C. W. Trigg, Los Angeles City College. Let the



vertices of ABC be $A(0,0)$, $B(a,0)$, $C(b,c)$, and the vertices of $A'B'C'$ be A' $[a - p(a - b), pc]$, B' $[(1 - m)b, (1 - m)c]$, C' $[na, 0]$. Then let the lines AA', BB', CC' be divided in the ratio r by A'' $\{r[a - p(a - b)], prc\}$, B'' $\{a - r[a - (1 - m)b], r(1 - m)c\}$, C'' $\{b - r(b - na), (1 - r)c\}$. Now denote the areas of ABC, $A'B'C'$, and $A''B''C''$ by S, K, and S'', respectively.

Then we have

$$K = \frac{1}{2}\begin{vmatrix} a - p(a - b) & pc & 1 \\ (1 - m)b & (1 - m)c & 1 \\ na & 0 & 1 \end{vmatrix} = \frac{ac}{2}\{1 - m - n - p + mn + np + pm\}$$

$$= S\{1 - m - n - p + mn + np + pm\}.$$

In like manner,

$$S'' = \frac{1}{2}\begin{vmatrix} r\ [a - p(a - b)] & pcr & 1 \\ a - r\ [a - (1 - m)b] & r(1 - m)c & 1 \\ b - r(b - na) & (1 - r)c & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} r\ [a - p(a - b)] & pca & 1 \\ r(1 - m)b & (1 - m)cr & 1 \\ rna & 0 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} r\ [a - p(a - b)] & 0 & 1 \\ r(1 - m)b & 0 & 1 \\ rna & (1 - r)c & 1 \end{vmatrix}$$

$$+ \frac{1}{2}\begin{vmatrix} 0 & pcr & 1 \\ a(1 - r) & (1 - m)cr & 1 \\ b(1 - r) & (1 - r)c & 1 \end{vmatrix}$$

$$= r^2 K + ac(r - 1)(2r - 1)/2 = r^2 K + S(r - 1)(2r - 1).$$

Hence, S'' is a constant since r, K, and S are constants.

AN INTEGER PROBLEM

153. March 1953 Proposed by P. A. Piza, San Juan, Puerto Rico.

(1) Find positive integers A, B such that $(A + B)^2 = 10^6 A + B$.

(2) Find positive integers C, D, n such that $(C + D)^{2n} = 10^6 C + D$.

Solution by C. W. Trigg, Los Angeles City College. (1) Let $B = k - A$, then $k^2 - k - 999{,}999A = 0$. Hence we seek values of A such that two factors of $999{,}999A$ differ by 1. We note that $A = 1{,}000{,}000$ leads to the trivial solution $B = 0$.

Now let the complementary factors of $999{,}999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$ be m and n, $m < n$. Then $n/m = p + r/m$. There is a smallest positive

solution, s_1, of $rs \equiv 1 \pmod{m}$ if m and n have no common factor, so $ns_1/m = q_1 + 1/m$. That is, $ns_1 = mq_1 + 1$ and $A = q_1 s_1$, $k = ns_1$, $B = (n - q_1)s$, and $999{,}999A = mnq_1s_1$.

We also may find a smallest positive solution, s_2, of $rs \equiv -1 \pmod{m}$, whereupon $ns_2/m = q_2 - 1/m$, or $ns_2 = mq_2 - 1$. Then $A = q_2s_2$, $k = ns_2 + 1$, $B = (n - q_2)s_2 + 1$, and $999{,}999A = mnq_2s_2$. It follows from $rs_1 \equiv 1$, $rs_2 \equiv -1$, $r^2s_2{}^2 \equiv 1$, that $s_1 = s_2\,(rs_2 + 1) - s_2$ and $s_1 + s_2 \equiv 0 \pmod{m}$.
Hence $s_1 + s_2 = m$.

Furthermore, since $q_1 = ps_1 + (rs_1 - 1)/m$ and $q_2 = ps_2 + (rs_2 + 1)/m$, it follows that $B_2 - B_1 = n(s_2 - s_1) + (p + r/m)\,(s_1{}^2 - s_2{}^2) - (s_1 + s_2)/m + 1 = m(s_2 - s_1)\,(n/m - p - r/m) = 0$.
Hence $B_2 = B_1$.

Thus for each set of values of m, n, r, B there are two sets of values for s, q, A. The thirty possible solutions of (1) follow:

m	n	r	s	q	A	B
27	37037	20	23	31550	725650	126201
			4	5487	21948	
189	5291	188	188	5263	989444	5264
			1	28	28	
297	3367	100	199	2256	448944	221089
			98	1111	108878	
351	2849	41	137	1112	152344	237969
			214	1737	371718	
999	1001	2	500	501	250500	250000
			499	500	249500	
7	142857	1	1	20408	20408	122449
			6	122449	734694	
77	12987	51	74	12481	923594	37444
			3	506	1518	
91	10989	69	62	7487	464194	217124
			29	3502	101558	
259	3861	14	205	3056	626480	165025
			54	805	43470	

m	n	r	s	q	A	B
11	90909	5	9	74380	669420	148761
			2	16529	33058	
143	6993	129	51	2494	127194	229449
			92	4499	413908	
407	2457	15	190	1147	217930	248900
			217	1310	284270	
13	76923	2	7	41420	289940	248521
			6	35503	213018	
481	2079	155	90	389	35010	152100
			391	1690	660790	
37	27027	17	24	17531	420744	227904
			13	9496	123448	

(2) Consider $(C + D)^x = 10^6 C + D$ where $x > 2$, and let $D = k - C$. Then $k(k^{x-1} - 1) = 999{,}999C$. Let $999{,}999 = mn$ and $C = yz$. Now $k = C$ leads to the trivial result $D = 0$, and if $k = 999{,}999$ then $k^{x-1} - 1 = C$. But this cannot be, since $k > C$. Hence $k = my$ and $k^{x-1} - 1 = nz$. Then since $k > C$, $m > z$. Thus for any complementary pair m, n an upper limit for $nz + 1$ is easily established, as well as a lower limit for k or my. Within these bounds it quickly becomes evident that no solution for C, D exists for $x > 2$.

Also solved in part by Leon Bankoff, Los Angeles, California and the proposer.

KEYS TO FOUR DOORS

154. [March 1953] Proposed by J. M. Howell, Los Angeles City College.

There are four doors and four keys, each of which fits one and only one door. What are the probabilities that all the doors will be opened in exactly k trials ($k = 4, 5, \ldots, 10$), where the trying of a key in a lock is considered a trial?

Solution by W. W. Funkenbusch, Michigan College of Mining and Technology, Sault Ste. Marie Branch. Let us first postulate the following (S for Success, F for Failure):

Postulate 1: The last two trials must be S's
Postulate 2: Both the $(k - 3)$rd and $(k - 2)$nd trials can not be F's

Postulate 3: The only possible situation in which there could be three consecutive F's would be the first three trials.

Case $k = 4$: This obviously can happen in only one way (i.e.) four consecutive S's for which the probability is obviously $\frac{1}{4!}$ or $\frac{1}{24}$

As a prelude to the analysis of the following cases let us state that the number of ways that they could occur surely has an upper limit of $^{C}(k - 2,2)$. This follows from a consideration of postulate 1. In order to determine the number of ways that a certain value of k can exist, we are able then to set $^{C}(k - 2,2)$ as the upper limit and to reduce this number of ways by the use of postulates 2 and 3.

Case $k = 5$: $^{C}(3,2) = 3$ ways, none of which are eliminated by postulates 2 or 3, and each way may be seen to have an existence probability of $\frac{1}{4!}$ therefore the probability that all four doors will be opened in exactly five trials is $\frac{3}{4!}$ or $\frac{1}{8}$

Case $k = 6$: $^{C}(4,2) = 6$ways, of which the way SSFFSS is eliminated by postulate 2. Each of the five non-eliminated ways may be seen to have an existance probability of $\frac{1}{4!}$ therefore the probability that all 4 doors will be opened in exactly 6 trials is $\frac{5}{4!}$ or $\frac{5}{24}$

Case $k = 7$: $^{C}(5,2) = 10$ ways, of which SSFFFSS is eliminated by both postulates 2 and 3, FSSFFSS and SFSFFSS by postulate 2, and SFFFSSS by postulate 3. The non-eliminated ways then total 6, each of which has an individual existance probability of $\frac{1}{4!}$ therefore the probability that all four doors will be opened in exactly 7 trials is $\frac{6}{4!}$ or $\frac{1}{4}$

Case $k = 8$: $C(6,2) = 15$ ways, of which by the use of postulates 2 and 3, we can eliminate all but 5, each of which has an individual existance probability of $\frac{1}{24}$ therefore the desired probability is $\frac{5}{4!}$ or $\frac{5}{24}$

Case $k = 9$: $C(7,2) = 21$ ways, of which, by the use of postulates 2 and 3, we can eliminate all but 3, each of which has an individual existance probability of $\frac{1}{4!}$ therefore the desired probability is $\frac{3}{4!}$ or $\frac{1}{8}$

Case $k = 10$: It is obvious that this could happen in only the following way FFFSFFSFSS, which way it is seen has an existance probability of $\frac{1}{4!}$

Also solved by Abraham L. Epstein, Boston, Mass.; H. F. Heller, Eastern Illinois State College; Sam Kravitz, East Cleveland, Ohio; Lawrence A. Ringenberg, Eastern Illinois State College; William Small, Rochester, New York; C. W. Trigg, Los Angeles City College, and the proposer.

A LOCUS OF ARC MIDPOINTS

155. [March 1953] Proposed by Leon Bankoff, Los Angeles, California.

A line segment is divided into two parts a and k - a. On each of these parts as a diameter a semicircle is drawn. Find the locus of the midpoint of the line composed of the arcs of the semicircles.

Solution by L. A. Ringenberg, Eastern Illinois State College.

Consider first the locus for a in the range $k/2 < a < k$. The total length of the two semicircles is $\pi k/2$. Point P is located so that arc $OP = \pi k/4$, arc $PQ = \pi(2a - k)/4$. Then $PR \cdot 2\theta = \text{arc } \overset{\frown}{PQ}$, $a\theta = \pi(2a - k)/4$, $\theta = (\pi/2) - (k\pi/4a)$, $0 \le \theta < \pi/4$, $a = k\pi/(2\pi - 4\theta)$, $r = a \cos\theta$, and

$$(1) \quad r = (k\pi\cos\theta)/(2\pi - 4\theta), \quad 0 \le \theta \le \pi/4.$$

Equation (1) is the polar equation of that part of the locus arising from a-values in the interval $(k/2, k)$. The points $S(k/2, 0)$ and $T(k/2^{\frac{1}{2}}, \pi/4)$ are the endpoints of the arc (1).

Along (1) we have

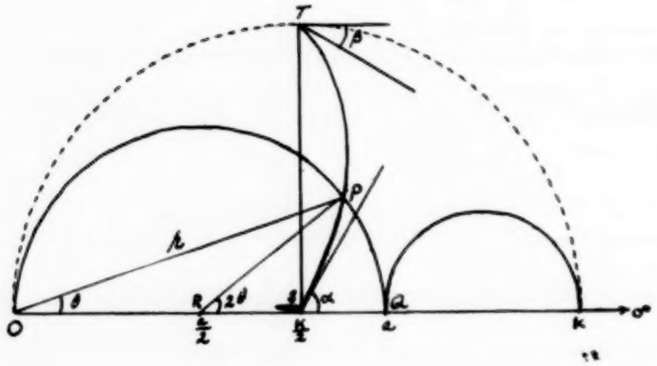
$$x - k/2 = r\cos\theta - k/2 = \frac{k}{2}\left[\frac{\pi\cos^2\theta}{\pi-2\theta} - 1\right] > 0$$

This last inequality follows from the relations

$$f(\theta) = \pi\cos^2\theta - \pi + 2\theta = 0 \quad \text{if } \theta = 0 \text{ or } \pi/4$$

$$f'(\theta) = -\pi\sin 2\theta + 2$$

$$f''(\theta) = -2\pi\cos 2\theta < 0 \quad \text{if } 0 < \theta < \pi/4$$



Therefore arc (1) joins S and T and lies to the right of the segment ST. The desired locus for a-values in the interval $(0, k/2)$ is the reflection of arc (1) across the segment ST. The total locus is either a smiple closed curve with corners at S and T or an S-shaped curve without corners if semicircles are on opposite sides of the segment. To aid in sketching we determine the tangents at S and T using elementary calculus: $\tan\alpha = \pi/2$, $\tan\beta = 2/(\pi - 2)$.

Also solved by C. W. Trigg, Los Angeles City College and the proposer.

MAXIMUM SEGMENTS IN AN ELLIPSE

166. [March 1953] Proposed by W. B. Carver, Cornell University.

What is the maximum length of a line segment that can be drawn in the smaller segment of the ellipse $b^2x^2 + a^2b^2 = a^2b^2$ cut off by the line $x = h$, $a > b$, $0 \leq h < a$?

Solution by the proposer. Let the line $x = h$ cut the ellipse at the points $A(h,c)$ and $B(h,-c)$, where

$$(1) \qquad c = \frac{b\sqrt{a^2 - h^2}}{a} > 0$$

It is rather obvious that the required maximum line segment must be a chord with one end at A or B, say at B, and the other end at some point P on the minor arc AB. The problem then is to find the maximum length of the chord BP as P moves over the minor arc AB from A to B. It will be convenient to use as a parameter the reciprocal s of the slope of the chord BP. The slope of the tangent to the ellipse at B is b^2h/a^2c or $bh/(a\sqrt{a^2 - h^2})$; and if we let

$$s_0 = \frac{a^2c}{b^2h} = \frac{a\sqrt{a^2 - h^2}}{bh},$$

we see that s varies over the interval $0 \leq s \leq s_0$ as P runs from A to B.

We first find the variable length λ of the chord BP as a function of s. The equation of the chord is

$$s(y + c) = x - h.$$

Solving this equation with the equation of the ellipse, and discarding the solution $x = h$, $y = -c$, we find the coordinates of P to be

$$\left[\frac{a^2h + 2a^2cs - b^2h\,s^2}{a^2 + b^2s^2}, \frac{a^2c - 2b^2hs - b^2cs^2}{a^2 + b^2s^2}\right];$$

and we then have

$$\lambda^2 = \left[\frac{a^2h + 2a^2cs - b^2hs^2}{a^2 + b^2s^2} - h\right]^2 + \left[\frac{a^2c - 2b^2hs - b^2cs^2}{a^2 + b^2s^2} + c\right]^2$$

$$= \frac{4(a^2c - b^2hs)^2(1 + s^2)}{(a^2 + b^2s^2)^2},$$

or

$$(2) \qquad \lambda = \frac{2(a^2c - b^2hs)\sqrt{1 + s^2}}{a^2 + b^2s^2}$$

We consider the variation of this function $\lambda(s)$ over the interval $0 \le s \le s_0$. It is evident that $\lambda(0) = 2c$ and $\lambda(s_0) = 0$. Differentiating we have

$$D_s\lambda = -\frac{2\{a^2b^2cs^3 + b^2h(2a^2 - b^2)s^2 - a^2c(a^2 - 2b^2)s + a^2b^2h\}}{(a^2 + b^2s^2)^2\sqrt{1 + s^2}},$$

and this derivative vanishes when

$$(3) \qquad a^2b^2cs^3 + b^2h(2a^2 - b^2)s^2 - a^2c(a^2 - 2b^2)s + a^2b^2h = 0.$$

Case I, in which $a^2 \le 2b^2$ may be disposed of at once. In this case all the coefficients in equation (3) are positive (or zero), and $D_s\lambda$ is therefore negative for all values of s in the interval under consideration. Thus for ellipses nearly circular ($a^2 \le 2b^2$), whatever the value of h, the maximum chord from B is the bounding chord BA of length $\lambda(0) = 2c$.

In what follows we assume that $a^2 > 2b^2$. In equation (3) the coefficient of s is negative and all the others are positive, and hence there is just one negative root s_1 which has no significance for our problem. The discriminant of the cubic equation is (omitting the positive factor $4a^4b^6$)

$$\Delta = a^4(a^2 - 2b^2)^3 - (a^2 + b^2)^3(a^2 - b^2)h^2.$$

This discriminant is negative, zero, or positive according as h^2 is greater than, equal to, or less than k, where

$$(4) \qquad k = \frac{a^4(a^2 - 2b^2)^3}{(a^2 + b^2)^3(a^2 - b^2)}$$

We consider two cases:

Case II, $h^2 \ge k$; equation (3) has two imaginary roots or a positive double root.

Case III, $h^2 < k$; equation (3) has two positive roots, $s_2 < s_3$, or, when $h = 0$, the root $s_2 = 0$ and a positive root s_3.

In case II, $D_s\lambda$ is negative throughout the interval $0 \le s \le s_0$ (except that it is zero at one point when $h^2 = k$), and hence λ decreases

continuously as s increases and the maximum chord from B is the bounding chord BA of length $\lambda(0) = 2c$, as in case I.

In case III, the two positive roots of equation (3), s_2, s_3, are both in the interval $0 \leq s \leq s_0$, and we have

$$D_s\lambda < 0 \text{ for } 0 < s < s_2,$$
$$D_s\lambda > 0 \text{ for } s_2 < s < s_3,$$
$$D_s\lambda > 0 \text{ for } s_2 < s < s_0.$$

It remains then to determine whether the maximum value of λ is $\lambda(0)$ or the relatively maximum value $\lambda(s_3)$. We first find the condition under which these two values of λ are equal. For this we must have

$$\lambda(s_3) = 2c, \tag{5}$$

and also equations (1), (2), and (3) must be satisfied, where s in these equations means the critical value s_3. From (1) and (3) we deduce

$$h = \frac{a^2\{(a^2 - 2b^2) - b^2s^2\}}{(a^2 + b^2s^2)\sqrt{a^2s^2 + b^2}}, \quad c = \frac{b^2\{a^2 + (2a^2 - b^2)s^2\}}{(a^2 + b^2s^2)\sqrt{a^2s^2 + b^2}}; \tag{6}$$

and putting these values of h and c in (2), we have

$$\lambda = \frac{2a^2b^2(1 - s^2)^{3/2}}{(a^2 + b^2s^2)\sqrt{a^2s^2 + b^2}}.$$

Then equation (5) gives us

$$a^2(1 + s^2)^{3/2} = a^2 + (2a^2 - b^2)s^2$$

or

$$s^2\{a^4s^4 - (a^4 - a^2b^2 + b^4)s^2 - a^2(a^2 - 2b^2)\} = 0.$$

Since s_3 cannot be zero, we must have

$$s_3^2 = \frac{a^4 - 4a^2b^2 + b^4 + \sqrt{(a^4 - 4a^2b^2 + b^4)^2 + 4a^2(a^2 - 2b^2)}}{2a^4}$$

$$= \frac{a^4 - 4a^2b^2 + b^4 + (a^2 - b^2)\sqrt{(a^2 - b^2)(5a^2 - b^2)}}{2a^4}.$$

This value of s_3^2 put into equation (6) gives us

$$h^2 = \frac{a^2\{(a^2-b^2)(2a^4+a^2b^2+17b^4)-b^2(5a^2-b^2)\sqrt{(5a^2-b^2)(a^2-b^2)}\}}{2(a^2+3b^2)^2(a^2-b^2)} = g.$$

It may then be verified (though the details are rather long and tedious) that $\lambda(s_3) \lesseqgtr 2c$ according as $h^2 \gtreqless g$. Hence we have the following results for case III:

when $h^2>g$, the maximum chord from B is the bounding chord BA;

when $h^2<g$, the maximum chord from B is the chord BP with slope $1/s_3$;

when $h^2 = g$, the bounding chord BA and the chord BP with slope $1/s_3$ are equal maximum chords from B.

Since it may also be verified that

$$\begin{aligned} &\text{when } a^2 > 2b^2, \quad 0 < g < k; \\ &\text{when } a^2 = 2b^2, \quad 0 = g = k; \\ &\text{when } a^2 < 2b^2, \quad k < g < 0; \end{aligned}$$

it follows that the results stated above for case III are correct also for cases I and II, and hence give the complete solution of the problem.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 100. Express $(a_1^2 + b_1^2)(a_2^2 + b_2^2)(a_3^2 + b_3^2)$ as the sum of two squares. [Submitted by M. S. Klamkin.]

Q 101. Find integer solutions of $x^2 + y^2 = z^3$. [Submitted by M. S. Klamkin.]

Q 102. Prove that no three lattice points of a square lattice form vertices of an equilateral triangle. [Submitted by Leo Moser.]

ANSWERS

A 100. $$(a_1 + ib_1)(a_2 + ib_2)(a_3 + ib_3)$$
$$\equiv a_3(a_1a_2 - b_1b_2) - b_3(a_1b_2 + a_2b_1) + i\ [b_3(a_1a_2 - b_1b_2) + a_3(a_1a_2 + a_2b_1)]$$

Equating the modules of the two sides of the identity we have
$$(a_1^2 + b_1^2)(a_2^2 + b_2^2)(a_3^2 + b_3^2) = [a_3(a_1a_2 - b_1b_2) - b_3(a_1b_2 + a_2b_1)]^2$$
$$+ [b_3(a_1a_2 - b_1b_2) + a_3(a_1b_2 + a_2b_1)]^2$$

A 101. In the result of A 100 let $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$, whence we have
$$(a^2 + b^2)^3 = (a^3 - 3ab^2)^2 + (3a^2b - b^3)^2.$$
Hence a general solution is $x = a^3 - 3ab^2$, $y = 3a^2b - b^3$, $z = a^2 + b^2$, where the parameters a, b are integers.

A 102. The area of an equilateral triangle of side a is $\sqrt{3}\ a^2/4$, which is irrational. But by Q 54, page 173, January-February 1952, the area of a triangle with three points of a square lattice as vertices is rational. Hence this triangle cannot be equilateral.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

INTRODUCTION TO THE THEORY OF STATISTICS. By Victor Goedicke Harper and Brothers, New York, 1953, 286 pages, $4.50.

This book is a readable presentation of the basic theory behind the most frequently used branches of statistics, together with applications. In addition to classroom use, it is intended for people who use statistics and wish to learn more about the theory. It differs from other textbooks in that (1) the style is less formal, (2) the theoretical treatments require only algebra, and (3) it presents a fresh approach to the logical basis of correlation theory. The final chapter, entitled "Statistics and Common Sense", is a non-mathematical discussion of the pitfalls which arise in practical applications.

Ohio University — Victor Goedicke

A HIGH LEVEL TREATMENT OF THE MATHEMATICS BEHIND PROBABILITY THEORY - "STOCHASTIC PROCESSES". By J. L. Doob of the University of Illinois was published in March '53 by John Wiley & Sons, 654 pp., $10.00.

The lack of published background material in this field has led Dr. Doob to supply elementary definitions and theorems in detail. Then, he goes on to develop the mathematical basis of his subject with the results stated in probability language.

The processes covered include: processes with mutually independent random variables; processes with mutually uncorrelated or orthogonal random variables; Markov processes - discrete and continuous parameter; and martingales. Processes with independent and orthogonal increments, stationary processes — discrete and continuous par -

ameter; and martingales. Processes with independent and orthogonal increments, stationary processes - discrete and continuous parameter, and linear least squares prediction - stationary (wide sense) processes are also covered.

John Wiley & Sons Richard Cook

FINITE DEFORMATIONS OF AN ELASTIC SOLID. By F. D. Murnaghan.

In this short text, the author furnishes many interesting results in the finite deformation of elastic solids as well as an elegant and unified theory. Some of this theory has been previously developed by the author and others in published papers and other parts seem to be completely new. Though the principal tool is matrix theory, a reader with a fairly elementary knowledge of this subject should be able to follow the arguments of the text. The problems contain many hints as to solution and often furnish additional information on the theory.

There are three parts to the text. First, the author furnishes an introduction to those topics in the mathematical theory which are used in the remainder of the text. Then, the development of the elasticity theory is given. Finally, applications are made to several problems in finite deformation.

The matrix concept is introduced by considering functional relations and their Jacobians. Further, the following formal properties of matrices are discussed: addition and multiplication, the transpose and inverse, unitary, orthogonal Hermetian and symmetric matrices. Many of these topics are treated in the problems. Of particular interest and later use are: the treatment of the matrix element of area (the counterpart of the bivector in tensor analysis), and the reduction of a linear operator (a second order tensor under the unitary group) to canonical form.

Two methods of specifying strain are used: (1) in terms of the Cartesian orthogonal coordinates of the initial or undeformed state; (2) in terms of the Cartesian orthogonal coordinates of the final or deformed state. The strain matrix is defined in terms of the change in arc length due to the deformation. It is shown that this matrix is sensitive to rotations of the initial reference frame but insensitive to rotations of the final reference frame. Hence, principal directions of this matrix exist only for the initial reference frame. When the initial and final reference frames coincide, one may introduce a displacement vector relating the corresponding positions of points of the elastic solid. Then, the previous theory furnishes the well known formulas for the components of the strain

tensor. Finally, the three basic invariants of the strain matrix and the compatibility relations are considered. The author's treatment of this last topic is equivalent to discussing one aspect of the fundamental form problem: "when is a quadratic form reducible to a sum of squares?" It is known that the necessary and sufficient condition is that the curvature tensor associated with the original quadratic form vanish. In the author's development, only the elements of tensor analysis are used and it is shown that this condition is necessary.

The stress matrix is of necessity associated with the final state of the elastic medium and is defined in the usual manner. By defining the matrix of a virtual deformation in terms of the Cartesian coordinates of the final state, and using the principle that the virtual work of the body and surface surfaces must vanish for any virtual displacement which is a rigid body motion, the author determines the equilibrium relations and the equations of motion. Further, it is shown that the stress matrix is symmetric. The expression for virtual work is studied for the case of a finite deformation as well as for an infinitesimal deformation. To obtain the energy relation, the author introduces the mass density of the energy of deformation (an invariant of the strain matrix) and requires that the virtual work done by the surface and body forces in any virtual deformation shall equal the variation of the energy of deformation. This formulation of the energy relation is shown to lead to the stress-strain equations. Since these relations may be needed in determining boundary conditions in terms of the initial state variables, the author determines a modified stress matrix.

Chapters four and five deal with various expressions for the energy of deformation and hence lead to various possible stress-strain relations. If the expression in terms of the strain matrix for energy is invariant under a given rotation of the reference frame, the medium is said to be elastically insensitive to this rotation. Isotropic media are elastically insensitive to all rotations. By use of such a characterization, the author determines an expression for the energy of an isotropic medium. The following interesting result is obtained: no elastic medium which is initially in a state of stress other than that due to a pressure or tension can be isotropic. The linear and second order approximations for the elastic constants are determined for the general strain matrix. This theory is used in comparing the theoretical results with Bridgman's experimental work. Finally, by requiring that the elastic medium be insensitive to specified rotations, the relations among the elastic constants for a non-isotropic medium are determined. The constants of the first, second, and third degree are considered.

The final chapters are concerned with the application of the previous theory to the following problems: (1) homogeneous shear and tension in isotropic and non-isotropic media; (2) the compression of an isotropic spherical shell and circular tube; (3) the torsion of an isotropic circular cylinder. In the first class of problems, the stress matrix is determined algebraically; in the last two classes of problems, it is necessary to solve ordinary differential equations.

N. Coburn

ANALYTIC GEOMETRY. By Raymond D. Douglass and Samuel D. Zeldin, McGraw-Hill Book Company, New York, 1950, ix and 216 pages, $2.75.

This book presents in simple and teachable form the elementary topics of both plane and solid analytic geometry. It develops the principles necessary for further study in differential and integral calculus without loss of time on non-essentials. There are many examples that illustrate the applications of the principles involved to other sciences as well as mathematics, and the student will find numerous exercises that furnish ample practice in the techniques to be learned. Review exercises at the end of each chapter should help the student properly to organize and coordinate the material already studied; answers are also provided.

The book is direct in approach; in the first chapter it moves easily through fundamental notions to distance between two points and area of a triangle. The second chapter deals with the straight line: slope, angle between two lines, conditions for parallelism and perpendicularity, as well as the point-slope, two point and intercept forms of equation of a straight line. One of the most unusual features of the book is the way in which the authors obtain the formula for distance from a line to a point. Starting with the general equation $Ax + By + C = 0$, they arrive at the statement that $h = \pm (Ax_1 + By_1 + C)/\sqrt{A^2 + B^2}$ without mentioning the normal form of straight line equation which is so often a real stumbling block to the student. They say the sign in the right-hand member of the formula "depends on the position of the point P, relative to the line L", without explaining fully about the sign. Would it not clarify the situation to tell the student how to determine the proper sign of h and what it indicates?

Chapter 3 studies the circle starting with standard form and moving on to the general equation and to circles determined by

three conditions. The authors give a formula for the length of the tangent from a fixed point to a circle as well as an equation for the tangent to a circle at a fixed point.

Translation of coordinate axes is introduced in chapter 4 to facilitate the discussion of the ellipse with axes parallel to the coordinate axes. There is a helpful reminder in the discussion of hyperbolas in chapter 5 concerning the asymptotes for a hyperbola with center at (h, k). Chapter 6 is a study of the parabola and points out an interesting fact not usually stated in textbooks that "the squares of any two chords perpendicular to the axis of a parabola are proportional to their distances from the vertex of the parabola".

Chapter 7 opens with the derivation of the formula for the rotation of axes through an acute angle. Then, starting with the general equation of second degree the authors show by an illustrative example how to eliminate the xy term by rotation; they also state without proof the relation between the coefficients of the general equation for specific conics. Figure 52 on page 87 shows the various conic sections.

Chapters 8 through 11 include many illustrative examples with numerous graphs of representative curves of each type. Chapter 8 studies algebraic curves; the discussion includes many excellent sketches. Chapter 9 deals with trigonometric and exponential curves: tables of trigonometric functions, common and natural logarithms are given at the end of the book for the student's convenience. Chapter 10 on polar coordinates includes many graphs as well as problems in deriving the polar equation directly from the given properties of the locus. In chapter 11 the authors discuss parametric representation of curves, draw figures of the trochoid and various cycloids, and give the names of many additional curves in the exercises at the end of the chapter.

Chapters 12 and 13 are devoted to solid analytic geometry. Distance between two points, direction cosines, angle between line segments, equations of a plane and of a straight line are fully covered, with numerous examples for each topic. In chapter 13 cylindrical surfaces, surfaces of revolution and quadric surfaces are presented with accompanying figures, and an introduction to cylindrical and spherical coordinates is offered.

This book seems to cover very well the topics essential to preparation for a course in calculus. It is remarkably concise but seems to be no less thorough for its brevity.

Wellesley College Vivian Gummo

GENERALITES SUR LES PROBABILITES. By Fréchet Maurice, Elements Aleatoires, Gauthier-Villars, Paris, 1950, xvi plus 355 pp., paper-bound.

This is the second edition of the first of two books planned by Frechet on "Modern Theoretical Research in the Calculus of Probability". These two books in turn form part of a "Treatise on the Calculus of Probability and its Applications" prepared by Emile Borel. As a part of a larger work, the theoretical material is developed in detail with illustrative examples, but applications, discussed elsewhere in the treatise, are given minimum attention.

Chapter one develops the notion of probability from several points of view, including equally probable events, frequency, chance and axiomatic.

Chapter two generalizes the theorem that the probability of at least one of a finite number of incompatible events is the sum of the probabilities of the individual events to the case where the events are compatible and also infinite in number. Since these formulas become quite complicated, certain useful inequalities are also derived.

Chapter three discusses typical values of a stochastic number, X. For example, a typical value of X of order r is the value of a which minimizes the expected value of $|X-a|^r$. It is shown that the mean, median, mode and center of a distribution are typical values of orders 2, 1, 0, and ∞, respectively. The rth root of the expected value of $|X- \ |^r$ is called the error of order r. More general definitions of typical values are also discussed. Probabilities associated with repeated trials of an event, and a discussion of characteristic functions, and other functions which represent a probability law conclude the chapter.

In Chapter four, Bienaymé's inequality is generalized, including Tchebycheff's inequality and others using errors of various orders other than the standard error of order two. When unimodal density functions only are considered, inequalities much sharper than Bienaymé's are obtained. These inequalities are shown to hold for even more general distribution functions than those with unimodel densities.

Chapter five discusses the convergence of sequences of stochastic numbers. Corresponding to each type of convergence, such as Cauchy convergence, or convergence in probability, an abstract space can be developed. Hence the already known theory of abstract spaces can be applied.

Chapter six treats of generalized stochastic variables, such as functions or curves, which are not simple stochastic numbers.

The ideas of typical values, distribution functions, and convergence for such stochastic variables are presented.

The reader is struck by the wide variety of topics which are brought together as special examples of more general theories, by the fertile generalizations of elementary notions, and by the cleverly contrived examples which illustrate the need for rigorous statements of theorems. While most of the work is naturally drawn from others, much of it is originally due to Frechet. Being written in treatise style, with all the fine points included, the language is straightforward, of average difficulty. A few typographical errors require the reader tocheck the formulas carefully. Regretably, there is no index. While the work is developed from first principles and includes a number of elementary results, the reader needs to be fairly sophisticated mathematically. Stieltjes integrals are used throughout, and an acquaintance with elementary mathematical statistics is adviseable. An excellent bibliography mostly to non-English languages and journals, is included.

The author has brought together a vast amount of material and unified it in an effective way, tying many of the problems into the modern theory of abstract spaces. This work will be a real aid to all serious workers in probability and statistics.

Paul B. Johnson

THE ANATOMY OF MATHEMATICS. By R. B. Kershner and L. R. Wilcox, The Ronald Press Company, New York, N. Y., 1950, xi + 416 pp. $6.00.

As stated in the preface, "this work has been produced in the hope that students may be aided in bridging the gap between classical and modern approaches, and that the terminologies and points of view which the axiomatic method entails may become more readily accessible to those who suddenly find themselves in need of becoming familiar with them" ..."it is hoped that at least a few so-called laymen will take advantage of the opportunity, here provided, to learn what modern mathematics is like, without being expected to bring an elaborate technical education to lay upon the altar. The only prerequisites for reading this book are the desire to start and the perseverance to finish. The reader does not even need to know the sum of 7 and 5; incidentally, if he does not know this sum, he will not learn it from this book."

It seems doubtful whether the book will accomplish this purpose, chiefly because, although not depending upon the technical aspects of mathematical training, it presupposes greater maturity than is, at

present, usually possessed by the readers to whom it is addressed. Unquestionably it would be very desirable for such persons to reach the understanding of the character of mathematics which this book makes possible. It sets an important aim.

After three introductory chapters and two on the materials of mathematics (sets, relations, operations), there follow chapters devoted to the postulational method, groups, the positive integers, finite sets, inductive definition and the principle of choice, extended operations, infinite sets, isomorphism and categorical systems, equivalence and order relations, positive rationals, 1-dimensional continua, real numbers and fields. Of these, chapters 16-18 (rationals and reals) seem the most interesting. The others do not bring much that is new to the mature reader.

Here and there the text is too repetitious (on p. 104 the axioms for the theory of positive integers are stated, on the adjoining page "we restate the foundations for positive integers"); and there are long arid stretches.

Repeatedly, non-mathematical considerations slip in, which reflect the authors' individual preferences. In passing frequently from purely intuitive discussions, in preparation for an abstract axiomatic treatment, it is perhaps unavoidable that such personal judgments as "we do not wish to include it in the logical basis" (p. 162), "our point of view is such that a principle is required" (p. 163), "in view of our firm belief in the consistency of these axioms" (p. 208), should creep in.

References to the literature do not occur in the text, but there is a group of 11 titles at the end as "suggestions for further reading". There are "projects" throughout the book, and suggestions and answers to them are found in the appendix (pp. 367-410).

It is to be hoped that this book will contribute to an appreciation of mathematics on the part of its teachers and its users.

Swarthmore Arnold Dresden

STATISTICAL PRESENTATION. By John H. Myers, Littlefield, Adams, and Co., 132 Beckwith Ave., Patterson 3, N. J., pp. 68. 75¢.

Ordinarily a statistics handbook is a kind of literary digest of elementary text material with much left unsaid. Statistical Presentation is a refreshing exception in which a pattern is set for good practice in presenting facts for practical business use. Tables and charts illustrate proper location of title, body, and

marginal scales. It is regretable that some charts have the vertical scale caption written vertically in the margin rather than at the top. The title is properly located above the table or chart with source and explanatory notes below.

Chapter I is an introductory statement of objectives and methods. Chapter II illustrates a General Table and a Special Table with proper explanations. A list of guiding principles is given. Chapter III illustrates three types of maps and follows with a variety of charts. Appendix A is a table of values and simple relative indexes. Appendix B gives a frequency table using four different types of classes. The mid-class values and frequency totals are omitted. A brief bibliography is included.

The illustrative general table shows adequate stubs and captions and includes totals for types of values of obvious interest for comparison. The special table is clearly distinguished from the general table to emphasize special purpose requirements.

Illustrative charts are grouped under the headings of maps, bars, circles, lines, and pictures. Cases of improper arrangement are compared with the correct form to emphasize better presentation. As a rule all words and numbers are written horizontally which avoids spinning the paper around to bring the writing into easy view. Suitable legends are used to distinguish between bars and lines so that the chart is not cluttered with much writing.

Some prefer that legends be placed at the lower right below the body of the chart. Illustrative cases for presenting averages and indexes would be a useful addition to the handbook. These forms of presentation would greatly improve the variety techniques found in many books and business reports.

University of Alabama C. D. Smith

FOUNDATIONS OF MATHEMATICS. By Raymond L. Wilder, John Wiley & Sons Inc., New York, N. Y., 1952, xiv pp. 305.

This book has grown out of a course which the author has given at the University of Michigan, for more than 20 years, to first year graduate students and to seniors who had majored in mathematics, in the belief that people "who were to base their life's work on mathematics" should acquire "some knowledge of modern mathematics and its foundations".

It is divided into two parts Part I (pp. 3 - 185) deals with the fundamental concepts and methods of mathematics and consists of the following seven chapters: The axiomatic method (pp. 3 - 22),

Analysis of the axiomatic method (pp. 23 - 51), Theory of sets (pp. 52 - 77), Infinite sets (pp. 78 - 109), Well-ordered sets: ordinal numbers (pp. 110 - 133), The linear continuum and the real number system (pp. 134 - 157), Groups and their significance for the foundations (pp. 158 - 188). Part II is concerned with the development of various points of view on the foundations of mathematics. It contains five chapters: The early developments (pp. 189 - 208), The Frege-Russell thesis: mathematics an extension of logic (pp. 209 - 229), Intuitionism (pp. 230 - 249), Formalism (pp. 250 - 263), The cultural setting of mathematics (pp. 264 - 284). In the "Suggestions for use as a textbook" which follow the preface, the author says that "by judicious selection of material, a satisfactory one-semester course can be based on the book. The formal text is followed by an extensive bibliography (pp. 285 - 295) and by three indices (of symbols, of topics and technical terms, of names).

The structure of Part I follows the general pattern of a textbook; each of the seven chapters closes with a list of interesting problems. Part II is more nearly a set of lectures on various aspects of the foundations of mathematics.

The book is written in a lively, informal style, which certainly maintained the reviewer's interest and which is likely to carry along any reader who is at all interested in mathematics. The author says that "for mature students in other fields, such as philosophy", it is not necessary to make the calculus a prerequisite for his course, but that "no *MATHEMATICS* student" should take it "without having had calculus". It is the reviewer's judgement that, while general maturity is certainly helpful, the contents of this book can be properly appreciated only by persons who have had enough experience with the usual mathematical disciplines to make the abstract concepts which are introduced meaningful to them.

It is encouraging for teachers who have been working for a recognition of the significance of mathematics apart from its technical applications, that a place has been found for a course of this type and that the publication of this book has become possible.

The book has obviously been written with great care for clarity, for preciseness, for "logical" development. The few remarks which follow are intended to suggest the consideration of slight modifications when a revision of the book is undertaken, rather than to detract from its value.

That "All statements implied by an axiom system Σ hold true for all models of Σ "(p. 27 - 2.2.1) is obviously meant to refer to "consistent" systems Σ. Not all the statements implied by the system introduced in the first paragraph on p. 24, hold true for all its models.

It appears that the rule (3a) on p. 95 does not give an effective definition of the set S', since it requires, for every element x of S, the decision whether it is contained in the corresponding set S_x. Moreover, the proof of the theorem in *3.2.1.3* is not convincing, because there seems to be no reason to suppose that the set S_a (i.e. S') should contain the element a.

The term "non-degenerate" occurring on pp. 170 and 171 does not appear to have been defined: the same seems to be true of the symbol i on p. 259.

Arnold Dresden

FIRST COURSE IN PROBABILITY AND STATISTICS. By Jerzey Neyman, New York, Henry Holt & Co., 1950, ix plus 350 pp. $3.50

Striking a bold contrast with other elementary texts, Neyman centers this book on the testing of statistical hypothesis, or probability as a guide to inductive behavior. This relatively advanced idea is made the core of a one semester course by considering only discrete cases and omitting such topics as correlation, regression, chi-square and Student's distribution.

Chapter I, on the scope of the theory of probability and statistics, is beautifully written and should be read by all puzzled students of the subject. The concepts of probability are developed without reference to the idea of "equally likely" in a manner which will be confusing to some. A detailed study of heredity leads to the binomial, poisson and normal distributions and to the problems of statistical hypotheses.

It is pleasant to see a text which emphasizes that the problem of a statistician is the making of choices, not merely determining probabilities. Many problems illustrate how discriminating persons are aided by probability in reaching decisions, while very few of the common run and gambling problems are given. Maturity, rather than specific mathematical ability beyond algebra, is the prerequisite. While some of the reasoning is quite close, the student may well gain a better understanding of the purpose of statistics from this text than he would from the usual course where he might learn more techniques.

Paul B. Johnson

(Ctd. on page 78)

OUR CONTRIBUTORS

Leonard Carlitz, Professor of Mathematics, Duke University, was born in Philadelphia in 1907. Educated at the University of Pennsylvania (A.B. '27, A.M. '28, Ph.D. '30), he spent 1930-32 as a National Research Fellow at the California Institute of Technology and at Cambridge University, and joined the Duke faculty in 1932. He served ad Editor, Duke Mathematical Journal foom 1938-50. His main mathematical interest lies in the theory of numbers.

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(continued on back of table of contents)